

Twenty-ninth Annual Columbus State Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
March 1, 2003

The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one “best” answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, -3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 39, 41, 45, 46, 48 and 49.

Throughout the exam, \overline{AB} will denote the line segment from point A to point B and AB will denote the length of \overline{AB} . Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.**

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1. The ratio $\frac{3^{2002} \cdot 7^{2004}}{21^{2003}}$ simplifies to which of the following fractions?

- (A) $\frac{1}{3}$ (B) $\frac{1}{7}$ (C) $\frac{1}{21}$ (D) $\frac{3}{7}$ (E) $\frac{7}{3}$

2. Simplify the expression $\frac{\sqrt[6]{a^3b^2}}{\sqrt[8]{a^5b^3}}$ given that a and b are positive real numbers.

- (A) $\frac{a}{b}$ (B) $\frac{\sqrt{a}}{b}$ (C) $\sqrt[8]{a} \sqrt[24]{b}$ (D) $\frac{1}{\sqrt[8]{a} \sqrt[24]{b}}$ (E) $\frac{\sqrt[8]{a}}{\sqrt[24]{b}}$

3. A professor asked Mary to multiply a number by 7 and then to add 11. Instead, Mary multiplied the number by 11, then added 7 and got 40. What was the number that Mary was supposed to obtain?

- (A) 68 (B) 10 (C) 12 (D) 32 (E) 45

4. Find the slope of a line perpendicular to the line $6x + 9y + 5 = 0$.

- (A) $-\frac{2}{3}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $-\frac{5}{9}$ (E) $\frac{5}{6}$

5. Find the product of the roots of the equation $(2x - 3)(x + 1) + (2x - 3)(2x + 1) = 0$.

- (A) 1 (B) -1 (C) 2 (D) $\frac{-2}{3}$ (E) $\frac{3}{2}$

6. Find the solution of the equation $\frac{2x}{x-1} + \frac{3x-2}{x+1} = 5$.

- (A) $\frac{2}{3}$ (B) $\frac{1}{2}$ (C) $\frac{3}{5}$ (D) $\frac{7}{3}$ (E) $\frac{2}{5}$

7. Find the number of positive solutions of the equation $\sqrt{x^2 - 6x + 9} = 4$.

- (A) 4 (B) 2 (C) 1 (D) 3 (E) 0

8. Find $x + y$ if $2x + y = x + 2y = 1$.

- (A) $\frac{1}{2}$ (B) 1 (C) $\frac{2}{5}$ (D) $\frac{3}{2}$ (E) $\frac{2}{3}$

9. A certain two digit number is equal to twice the sum of its digits. Find the product of its digits.

- (A) 8 (B) 6 (C) 12 (D) 15 (E) 10

10. A concrete patio is to be built 18 feet long, 9 feet wide and 8 inches thick. Knowing that a foot equals 12 inches and a yard is 3 feet, how many cubic yards of concrete are needed?

- (A) 2 (B) 4 (C) 3 (D) 6 (E) 5

11. For the numbers a , b and c whose sum is not zero, define $(a, b, c) = \frac{2ab + 3bc + 4ac}{(a + b + c)^2}$. Find $(3, 2, 1)$.

- (A) $\frac{6}{5}$ (B) $\frac{5}{36}$ (C) $\frac{25}{6}$ (D) $\frac{5}{6}$ (E) $\frac{5}{3}$

12. Simplify the expression $(a^2 + b^2 + c^2 - ab - ac - bc)(a + b + c)$.

- (A) $a^3 + b^3 + c^3 - 3abc$ (B) $a^3 + b^3 + c^3 + 3abc$ (C) $a^3 + b^3 + c^3$
(D) $a^3 + b^3 + c^3 - abc$ (E) $a^3 + b^3 + c^3 - 2abc$

13. Find the sum of the number $N = 1234$ and all other four digit numbers obtained by permuting the digits of N in all possible ways.

- (A) 33330 (B) 999990 (C) 55550 (D) 66660 (E) 11110

14. If the base of a triangle is increased $66.\bar{6}\% = 66.666\dots\%$ and the altitude to this base is decreased 40% , then what is the change in the area of the triangle?

- (A) 1% increase (B) 0% increase (C) $\frac{1}{2}\%$ increase
(D) $\frac{1}{2}\%$ decrease (E) 1% decrease

15. For what value of the parameter m does the equation $\frac{x+2}{3x+1} = \frac{2x-1}{6x-m}$ have no solution for x ?

- (A) 12 (B) 18 (C) 14 (D) 17 (E) 13

16. Find the value of x such that $4(1+y)x^2 - 4x + 1 - y = 0$ is true for all real values of y .

- (A) $\frac{-1}{2}$ (B) $\frac{-2}{3}$ (C) $\frac{2}{3}$ (D) $\frac{1}{2}$ (E) $\frac{3}{2}$

17. If $\log_{10} 2 = a$ and $\log_{10} 3 = b$ then find the expression which gives $\log_5 24$.

- (A) $\frac{3a+b}{1+a}$ (B) $\frac{3a-b}{1-a}$ (C) $\frac{3a+b}{1-a}$ (D) $\frac{2a+b}{1-a}$ (E) $\frac{a+2b}{1-a}$

18. If $\log_2(\log_2(\log_3 x)) = \log_2(\log_3(\log_2 y)) = 0$ then what is $x + y$?

- (A) 20 (B) 12 (C) 6 (D) 17 (E) 15

19. Knowing that 2 is a solution of the equation $6x^3 - 25x^2 + 32x - 12 = 0$, find the sum of the other two solutions.

- (A) $-\frac{12}{5}$ (B) $\frac{12}{5}$ (C) $-\frac{13}{6}$ (D) $\frac{13}{6}$ (E) $\frac{5}{6}$

20. The quadratic equation $3x^2 + 4x + 5 = 0$ has complex roots r and s . Determine a quadratic equation with roots $3r + 1$ and $3s + 1$.
- (A) $9x^2 - 18x - 12 = 0$ (B) $3x^2 + 6x - 30 = 0$ (C) $x^2 - 2x - 12 = 0$
(D) $2x^2 + 4x + 24 = 0$ (E) $x^2 + 2x - 12 = 0$
21. Consider the sequence $2, 2, 4, 6, 3, 2, 5, \dots$. For $n > 2$, the n -th term of this sequence is the remainder of the division by 7 of the sum of the two previous terms. Find the 2003-rd term.
- (A) 6 (B) 5 (C) 1 (D) 3 (E) 4
22. An equilateral triangle with sides of length x has the same area as a square with sides of length y . Find the ratio $\frac{y}{x}$.
- (A) $\frac{\sqrt[4]{3}}{3}$ (B) $\frac{\sqrt[4]{3}}{6}$ (C) $\frac{2\sqrt[6]{3}}{3}$ (D) $\frac{\sqrt[4]{6}}{3}$ (E) $\frac{\sqrt[4]{3}}{2}$
23. The state income tax where Wanda lives is levied at the rate of $t\%$ ($0 < t < 100$) of the first \$30,000 of annual income plus $(t + 1)\%$ of any amount over \$30,000. Wanda noticed that the state income tax she paid amounted to $(t + 0.25)\%$ of her annual income. What was her annual income?
- (A) \$40,000 (B) \$35,000 (C) \$50,000 (D) \$55,000 (E) \$45,000
24. How many integers between 1 and 2003 are divisible by 7 and 13 but are not divisible by 49?
- (A) 20 (B) 22 (C) 19 (D) 23 (E) 17
25. Let a and b be two complex numbers satisfying $\frac{a}{b} + \frac{b}{a} = \sqrt{3}$. Which of the following is a possible value for $a^6 + b^6$?
- (A) 7 (B) 1 (C) -1 (D) 2 (E) 0

26. Three consecutive positive integers in increasing order, say a , b and c , satisfy the equality $ac^2 + b^2 = 2003$. Find $2a + 4b - 5c$.

- (A) -5 (B) 5 (C) 1 (D) 2 (E) 10

27. A rectangle is divided into four sub-rectangles with areas 6, 9, 12 and x , as in the adjacent figure. Find x .

6	9
12	x

- (A) 18 (B) 20 (C) 24
(D) 16 (E) 15

28. John and Mary must mow a rectangular lawn which is 72 feet by 65 feet. They decide that each will mow a region having exactly half of the lawn's area. John starts by mowing around the outside of the lawn a strip equally wide on each side of the lawn. How wide should John's strip be?

- (A) 9 (B) 10 (C) 8 (D) 11 (E) 12

29. The expression $x^2 - y^2 - z^2 - 2yz + x + y + z$ can be factored as a product of two polynomials in x and y with integer coefficients. Which of the following is one of these factors?

- (A) $x + y - z$ (B) $x + y + z + 1$ (C) $x - y + z$
(D) $x - y + z + 1$ (E) $x - y - z + 1$

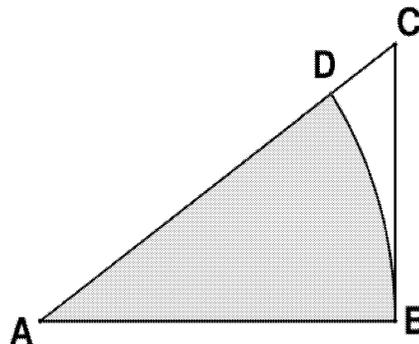
30. Let $h(x) = 6x^2 - 5x + 1$ be defined for all integer values of x . How many values of h are prime numbers?

- (A) 4 (B) 3 (C) 1 (D) 2 (E) 0

31. In the isosceles triangle $\triangle ABC$ ($AB = AC = 2$ cm) the angle bisector of the angle $\angle A$ and the altitude from B form an angle measuring 120° . Find the area of the triangle $\triangle ABC$ (in cm^2).

- (A) $\sqrt{3}$ (B) 4 (C) 3 (D) $\sqrt{2}$ (E) $\sqrt{5}$

32. In the accompanying figure, the right triangle $\triangle ABC$ has side lengths $AB = 4 \text{ cm}$ and $BC = 3 \text{ cm}$. The circle centered at A with radius AB intersects the hypotenuse \overline{AC} at D. Find the area of the shaded sector ABD (in cm^2) accurate to three decimal places.



- (A) 5.153 (B) 5.501 (C) 5.151
 (D) 5.148 (E) 5.203

33. Consider the function $g(x) = \sin x + 2 \cos x$ defined for all real values of x . Which of the following is not a possible value of g ?

- (A) $\frac{9}{4}$ (B) 1 (C) -2 (D) $\frac{-11}{5}$ (E) $\frac{13}{6}$

34. How many liters of a 40% solution of acid must be combined with a 15% solution to obtain 30 liters of a 20% solution?

- (A) 4 (B) 2 (C) 6 (D) 7 (E) 10

35. In a recent survey, 40% of occupied houses contained two or more people. Of those homes containing only one person, 25% contained a male. What is the percentage of all occupied houses which contain exactly one female and no males?

- (A) 30% (B) 40% (C) 45% (D) 50% (E) 55%

36. How many digits does the number 2^{2003} have?

- (A) 605 (B) 601 (C) 6002 (D) 603 (E) 604

37. If $\frac{1}{a} - \frac{1}{b} = 1$, find the value of $\frac{a - 2ab - b}{2a + 3ab - 2b}$.

- (A) 1 (B) -3 (C) 2 (D) -2 (E) -1

38. In the acute triangle ABC , $m\angle ABC = 45^\circ$, $AB = 3$ and $AC = \sqrt{6}$. Find the angle $\angle BAC$ in degrees.

- (A) 60° (B) 75° (C) 65° (D) 90° (E) 70°

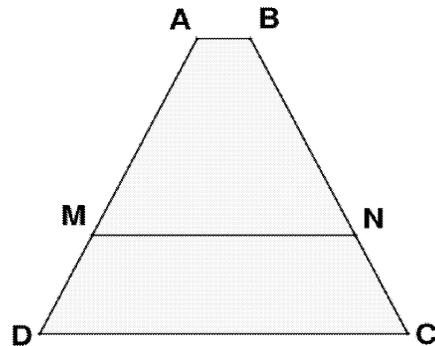
39. * Let f be a function from the positive real numbers into the positive real numbers. If $[f(xy)]^2 = x[f(y)]^2$ for all positive numbers x and y and $f(2) = 6$, find $f(50)$.

- (A) 40 (B) 30 (C) 50 (D) 10 (E) 20

40. If $a + b = 1$ and $a^2 + b^2 = 85$, find $a^3 + b^3$.

- (A) 127 (B) 173 (C) 154 (D) 321 (E) 543

41. * In the accompanying figure the quadrilateral $\square ABCD$ is an isosceles trapezoid of side lengths $AB = 1$ cm, $BC = DA = 5$ cm and $CD = 7$ cm. The points M on \overline{AD} and N on \overline{BC} determine a segment parallel with \overline{DC} which divides the area of the trapezoid into two equal parts. Find MN (in cm).



- (A) 5.4 (B) 4.5 (C) 4.7
(D) 5 (E) 5.2

42. Compute the sum of the arithmetic progression 7, 10, 13, 16, ..., 52.

- (A) 290 (B) 450 (C) 582 (D) 354 (E) 472

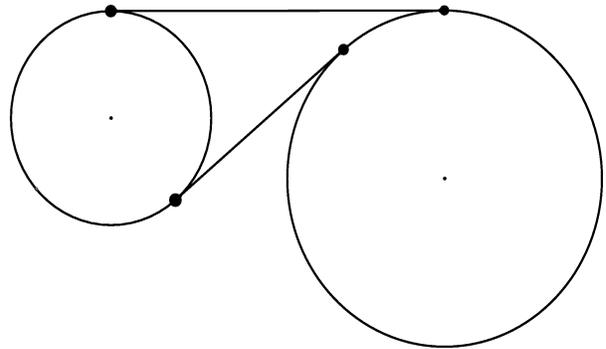
43. Let A be the angle in the first quadrant (measured in degrees) such that $\sin A = \frac{3}{5}$. Compute $\tan(360^\circ - A)$.

- (A) -0.15 (B) 0.75 (C) 0.55 (D) 0.15 (E) -0.75

44. If 4 people are seated in a random manner in a row containing 10 seats, what is the probability that the people will occupy 4 adjacent seats?

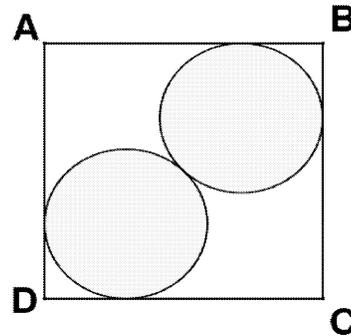
- (A) $\frac{1}{7}$ (B) $\frac{1}{50}$ (C) $\frac{1}{40}$ (D) $\frac{1}{30}$ (E) $\frac{1}{35}$

45. * The distance between the centers of two circles is 65 cm. The radii of these circles are 36 cm and 20 cm. There are two segments tangent to these circles (determined by tangency points), one longer than the other. Find the sum of these two segment lengths in centimeters.



- (A) 96 (B) 104 (C) 78
(D) 58 (E) 100

46. * In the accompanying figure, the quadrilateral $\square ABCD$ is a square with side lengths each equal to 2 cm. The two circles have the same radius length r , and they are tangent to each other and tangent to the sides of the square. Find r (in cm).



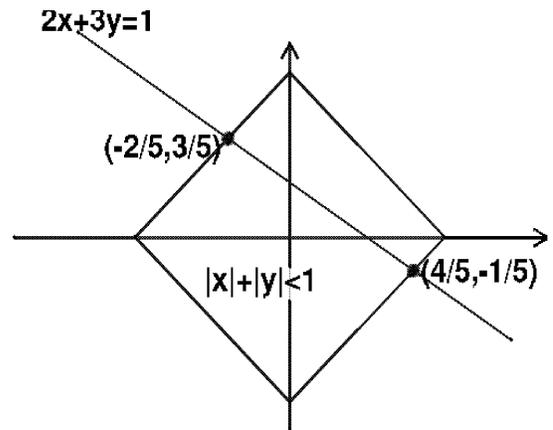
- (A) $\frac{7}{2} - 2\sqrt{2}$ (B) $\frac{2}{3}$ (C) $\frac{\sqrt{2}}{2}$
(D) $5 - 3\sqrt{2}$ (E) $2 - \sqrt{2}$

47. Several consecutive integers sum to 2003. Given that the number of these integers is an odd number greater than 3, determine the smallest of them.

- (A) -1050 (B) -1020 (C) -1001 (D) -1000 (E) -1002

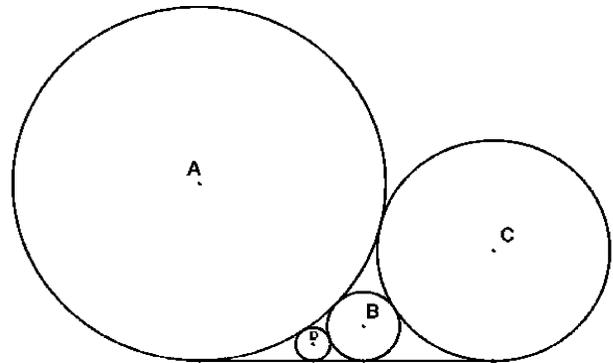
48. * A point $P(x, y)$ is randomly selected from the region $|x| + |y| < 1$. What is the probability that $2x + 3y < 1$?

- (A) $\frac{7}{10}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$
 (D) $\frac{3}{4}$ (E) $\frac{14}{10}$



49. * It is customary to refer to a circle centered at a point O and of radius r by $C(O, r)$. In the accompanying figure, circles $C(A, 9)$ and $C(B, 1)$ are externally tangent. Circles $C(C, r_1)$ and $C(D, r_2)$ are tangent to the previous two circles and all four circles are tangent to a line. Find the ratio $\frac{r_1}{r_2}$ ($r_1 > r_2$).

- (A) 5.5 (B) 5 (C) 6
 (D) 4 (E) 4.5



50. How many points with positive rational coordinates selected from the points in the xy -plane satisfy $x^2 + y^2 = 1$?

- (A) 1 (B) 2 (C) 2003
 (D) 100 (E) infinitely many