Thirtieth Annual Columbus State Invitational Mathematics Tournament

Sponsored by

The Columbus State University

Department of Mathematics

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The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 35, 37, 39, 40, 42, 43, 46 and 47.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet**.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1. If the arithmetic mean of two numbers is $5a$ and one of the numbers is $2a+3$, what is the other number?

   (A) $3a-3$ (B) $8a-3$ (C) $12a+3$ (D) $8a+3$ (E) $7a+3$

2. A fair coin is flipped three times. What is the probability that at least one head will be thrown?

   (A) 0.875 (B) 0.125 (C) 0.25 (D) 0.75 (E) 0.9

3. Each edge of a cube is increased by 2 inches so that the new cube's total surface area is 486 square inches. What was the length of the edge of the original cube?

   (A) 11 (B) 9 (C) 7 (D) 5 (E) 3

4. What is the value of $\left(\sqrt{3}^{\sqrt{3}}\right)^{\sqrt{3}}$?

   (A) 3 (B) $\sqrt{3}$ (C) $9\sqrt{3}$ (D) $3\sqrt{3}$ (E) $27\sqrt{3}$

5. A rectangular pool has a leak that causes the pool to drain at a rate of 0.3 cubic feet per minute. If the pool measures 12 feet wide by 18 feet long and 5 feet deep, how many hours will it take the water level in the pool to drop 1 foot?

   (A) 6 (B) 12 (C) 60 (D) 72 (E) 5

6. Let $x, y > 0$ and $A = \dfrac{1}{y}\left(1+\dfrac{1}{2y}\right)+\dfrac{1}{2y}\left(\dfrac{1}{x}-\dfrac{1}{y}\right)$ and $B = \dfrac{1}{2xy^2}\left(1+\dfrac{1}{2x}\right)$. Find $\dfrac{A^2}{B}$.

   (A) x (B) 1 (C) y (D) $2x+1$ (E) xy

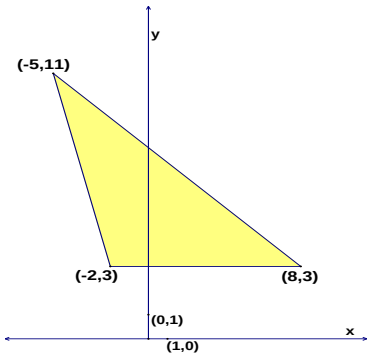
7. Four different circles are drawn on a sheet of paper. What is the largest possible number of points of intersection of these circles?

   (A) 12 (B) 14 (C) 8 (D) 6 (E) 10

8. Which of the following is the same as $\dfrac{3-6+9-12+15-18+21-...-2004}{7-14+21-28+35-42+49-...-4676}$?

(A) (B) 1 (C) 4 (D) -1 (E) $\dfrac{3}{7}$ $\dfrac{6}{7}$

9. What is the area (in square units) of a triangle formed by the points $(-5,11)$, $(-2,3)$, $(8,3)$ on the xy plane?



(A) 24 (B) 40 (C) 48

(D) 80 (E) 30

10. Let n be a positive integer such that $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{n}$ is an integer. Which of the following is *not* true?

(A) 2 divides n (B) 3 divides n (C) $n > 5$

(D) 6 divides n (E) n is a prime number

11. The sum of 9 consecutive positive integers is a perfect square. Which of the following is a possible value of this sum?

(A) 225 (B) 16 (C) 289 (D) 49 (E) 169

12. Find the sum $S = 1 + 2 + 2^2 + 2^3 + ... + 2^{100}$.

(A) $2^{102} - 1$ (B) 2^{101} (C) 2^{1002} (D) $2^{101} - 1$ (E) $2^{101} - 2$

13. Find the sum $S = 13 + 16 + 19 + 22 + ... + (3k + 10) + ... + 100$.

(A) 1582 (B) 1690 (C) 1695 (D) 1469 (E) 1921

14. Compute $\displaystyle\sum_{k=1}^{2004}\left(\frac{1}{5}k+1\right)$.

(A) 403806 (B) 403802 (C) 403805 (D) 403807 (E) 403808

15. Consider the function g defined by $g(x) = \begin{cases} 4x+\dfrac{3}{7} & \text{if } 0 \le x < \dfrac{1}{7} \\ \dfrac{x-\dfrac{1}{7}}{2} & \text{if } \dfrac{1}{7} \le x < 1. \end{cases}$

Calculate $g(g(g(0.4)))$.

(A) 0.5 (B) 0.1 (C) 0.3 (D) 0.2 (E) 0.4

16. To which of the following can the expression

$$E = (a+b+1)^3 - a^3 - b^3 - 3(a+b)(a+1)(b+1)$$

be simplified?

(A) 1 (B) -2 (C) -1 (D) 0 (E) 2

17. Each of the numerator and the denominator of the fraction $F = \dfrac{166...66}{666...64}$ has n digits. Find n knowing that $F = \dfrac{1}{4}$.

(A) 2 (B) undetermined (C) 3

(D) 4 (E) 5

18. If $0 < t < \frac{\pi}{2}$ and $\cos t = \dfrac{x}{1-x}$ for some real number x. Find $\tan t$ in terms of x.

(A) $-\dfrac{\sqrt{1-2x}}{x}$ (B) $\dfrac{\sqrt{1-2x}}{x}$ (C) $\dfrac{\sqrt{1-x^2}}{x}$

(D) $-\dfrac{\sqrt{1-x^2}}{x}$ (E) $\dfrac{\sqrt{1-2x}}{1-x}$

19. The expression $E = \cos x \cos(x-y) - \sin x \sin(x-y)$ can be simplified to which of the following?

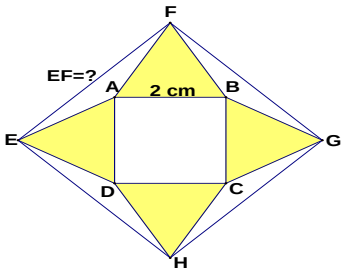
(A) $\sin x$ (B) $\cos y$ (C) $\cos(2x-y)$

(D) $\sin(x-y)$ (E) $\sin(x+y)$

20. The polynomial $P(x) = 12x^3 - 40x^2 + 27x - 5$ can be factored as $(3x-1)(ax^2+bx+c)$. Find $a+b+c$.

(A) -6 (B) -2 (C) -4 (D) -5 (E) -3

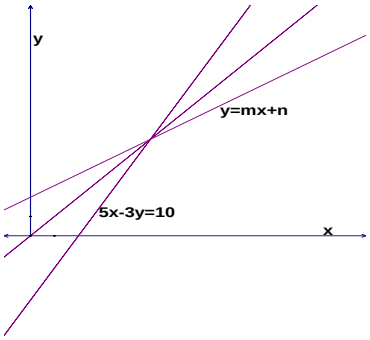
21. Equilateral triangles ABF, BCG, DCH, and ADE are constructed, as in the accompaning figure, on the sides of the square $ABCD$ (and in the exterior). This way a new square is formed: $EFGH$. If $AB = 2\ cm$, find EF in centimeters.



(A) $\sqrt{2}+\sqrt{3}$ (B) $\sqrt{2}+\sqrt{6}$ (C) $\sqrt{2}+\sqrt{12}$

(D) $\sqrt{8}+\sqrt{3}$ (E) $\sqrt{18}+\sqrt{3}$

22. The line $y = mx + n$ is the reflection of $5x - 3y = 10$ about the line $y = x$. Find n.



(A) 1 (B) 3 (C) -1

(D) 2 (E) -5

23. How many different real numbers satisfy the equation $(x^2+4x-2)^2 = (5x^2-1)^2$?

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

24. A shopper asked for 50 cents worth of apples. The shopper was surprised when she received five more apples than in the previous week (for the same price). Afterwards, she noticed that the price had dropped 10 cents per dozen. What was the new price of apples in cents per dozen?

(A) 50 (B) 35 (C) 20 (D) 40 (E) 30

25. The equation $(x-1)^5+(x-2)^4+(x-3)^3+(x-4)^2+(x-5)=0$ has five solutions (including complex numbers). What is their sum?

(A) 5 (B) 12 (C) 15 (D) 4 (E) 0

26. What is the value of the product below?

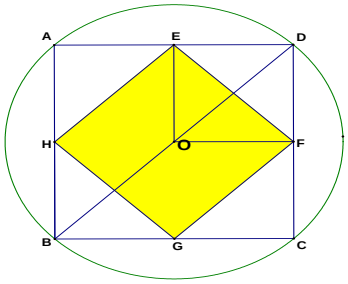
$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{2004^2}\right)$$

(A) $\frac{2002}{2003}$ (B) $\frac{2005}{4008}$ (C) $\frac{1}{2}$ (D) $\frac{2004}{2005}$ (E) $\frac{2001}{4008}$

27. The square $ABCD$ is inscribed in a circle of diameter 12 cm, and $EFGH$ has been created by joining the midpoints of each side of square $ABCD$. What is the area of $EFGH$ in square centimeters?

(A) 72 (B) 34 (C) 169

(D) 32 (E) 36



28. The cubic equation $x^3-12x^2+44x+a=0$ has three solutions in arithmetic progression. Find a.

(A) 50 (B) -48 (C) -55 (D) 20 (E) -35

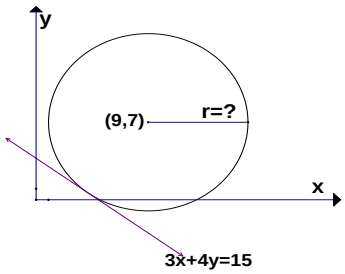
29. Find the remainder of the division $(x^4 - 5x^3 + 14x^2 - 15x + 10) \div (x^2 - 2x + 1)$.

(A) $x + 1$ (B) $2x + 1$ (C) $3x + 2$ (D) $2x + 3$ (E) 3

30. Find the radius of a circle with center $(9, 7)$ that is tangent to $3x + 4y = 15$.

(A) 10 (B) 7 (C) 8

(D) 9 (E) 6



31. Define $x \circ y$ to be x^y for all positive integers x and y. Which of the following is true for all positive integers x, y and z?

(A) $(x \circ y) \circ z = x \circ (y \circ z)$ (B) $(xy) \circ z = (x \circ z)(y \circ z)$ (C) $x \circ y = y \circ x$

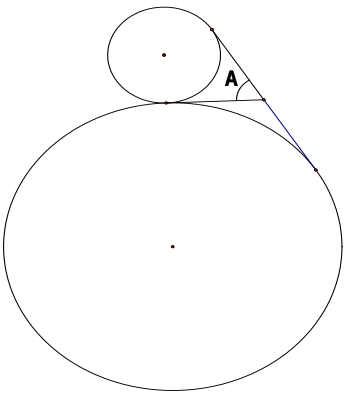
(D) $(2x) \circ y = 2(x \circ y)$ (E)

$x \circ (3y) = (3x) \circ y$

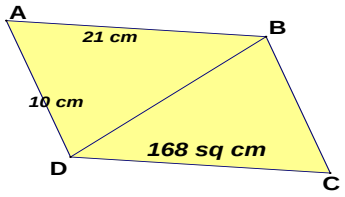
32. Two circles with radii measuring 3 inches and 9 inches are tangent externally (see figure). The two tangent lines to both circles intersect at an acute angle denoted here by A. Determine A in radians.

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{5}$ (C) $\frac{\pi}{12}$

(D) $\frac{\pi}{3}$ (E) $\frac{\pi}{6}$



33. The quadrilateral $ABCD$ has area 168 cm^2 and side lengths measuring $AB = CD = 21$ cm and $BC = DA = 10$ cm. Find the length of the shortest diagonal in centimeters.



(A) 17 (B) 15 (C) 18

(D) 19 (E) 12

34. If $1 < k < 10$ and $(\log_k x)(\log_7 k) = 3$, find x.

(A) k^2 (B) 343 (C) $21k$ (D) 125 (E) 2187

35. * The equation $2^x - 4x = 2004$ has only two real solutions. Which of the following best approximates their sum.

(A) -301 (B) -401 (C) -460 (D) -490 (E) -430

36. A right triangle has side lengths proportional to the numbers 3, 4 and 5 and an area of 726 square inches. How long is the hypotenuse (in inches)?

(A) 50 (B) 60 (C) 45 (D) 55 (E) 65

37. * Find the number of positive solutions of the equation $\sin x = \dfrac{x}{2004}$.

(A) 637 (B) 635 (C) 636 (D) 639 (E) 638

38. The equation $\dfrac{x-3}{2} - \dfrac{2}{x-3} = \dfrac{3}{x-2} - \dfrac{x-2}{3}$ has $x = 0$ as one of its solutions. Find the product of the other solutions.

(A) 17 (B) 16 (C) 15 (D) 14 (E) 13

39. * Let N=112233. How many different numbers (including N) are obtained by permuting the digits of N in all possible ways?

(A) 90 (B) 60 (C) 50 (D) 40 (E) 120

40. * Find all real numbers satisfying $\sqrt{1-\sqrt{1-x^3}} \geq x$.

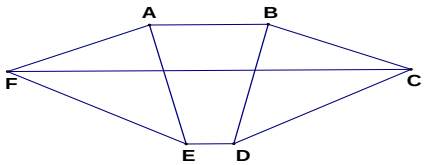
(A) $x \geq 0$ (B) $x = 0$ *or* 1 (C) $x \leq 1$

(D) $x = 1$ (E) $0 \leq x \leq 1$

41. Determine m such that the equation $x + 2|x + 1| + 2|x - 1| = m$ has exactly one real solution in x.

(A) 3 (B) 5 (C) 4 (D) 0 (E) 1

42. * In the accompanying figure two isosceles trapezoids, $ABCF$ and $FCDE$, share the same base $FC = 34\ in$ and their interiors do not overlap. Also $AB = 10\ in$, $ED = 4\ in$, $BC = AF = 13\ in$ and $DC = EF = 17\ in$. Find the area of the trapezoid $ABDE$ in square inches.



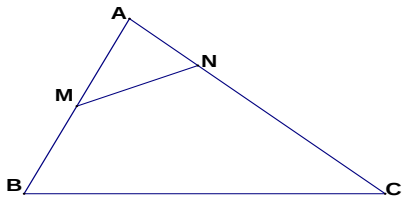
(A) 84 (B) 112 (C) 98

(D) 105 (E) 91

43. * How many integers x satisfy $(x^2 + x - 1)^{x-3} = 1$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

44. In the accompanying figure there is a triangle whose sides are given: $AB = 14\ in$, $BC = 18\ in$ and $AC = 21\ in$. The point M is the midpoint of $\overline{AB}$ and N is the point on $\overline{AC}$ for which $m\angle BMN + m\angle BCA = 180°$. Find MN in inches.



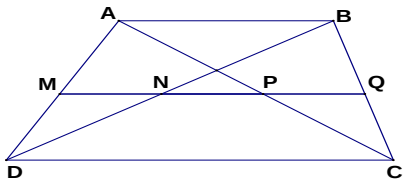
(A) 6 (B) 7 (C) 5

(D) 8 (E) 10

45. If a and b are positive numbers such that $a^{2b} = b^{3a}$ and $b = 5a$, find a.

(A) $5^{\frac{1}{7}}$ (B) $5^{\frac{2}{7}}$ (C) $5^{\frac{3}{7}}$ (D) $5^{\frac{4}{7}}$ (E) $5^{\frac{5}{7}}$

46. * The trapezoid $ABCD$ has the major base $DC = 6\ cm$ and the minor base $AB = 2\ cm$. Consider the points M on $\overline{AD}$ and Q on $\overline{BC}$ such that $\overline{MQ}$ is parallel to the bases. Also $\overline{MQ}$ is divided by the diagonals $\overline{AC}$ and $\overline{BD}$ into three equal parts ($MN = NP = PQ$, N is on $\overline{BD}$). Find the length of MN in centimeters.



(A) 1.3 (B) 1 (C) 1.2

(D) 1.5 (E) 1.7

47. * Knowing that every solution of the equation $3(\cos x)^2 - (\cos x) - 1 = 0$ is a solution of the equation $a(\cos 2x)^2 + b(\cos 2x) - 1 = 0$. What is $a + b$?

(A) 9 (B) 14 (C) 5 (D) 13 (E) 16

48. What is the minimum value of $f(x, y) = x^2 - 4x + y^2 + 6y$ when x and y are subjected to the restrictions $0 \leq x \leq 1$ and $0 \leq y \leq 1$?

(A) -1 (B) -2 (C) -3 (D) -5 (E) -6

49. Find the y-intercept of the tangent line to the parabola $y = x^2$ at the point $(1, 1)$.

(A) -5 (B) -4 (C) -3 (D) -2 (E) -1

50. Find $\lim_{x \to 0} \dfrac{3^x - 1}{2^x - 1}$.

(A) $\log_2 3$ (B) $\dfrac{\ln 2}{\ln 3}$ (C) 0 (D) 1 (E) $\ln(3/2)$