CSU High School Mathematics Tournament
March 6th, 2004
Ciphering Competition

Solutions

Round I

**Problem 1:** We first write the left side of the equation as a single fraction

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}} = \cfrac{1}{1+\cfrac{1}{\cfrac{x+1}{x}}} = \cfrac{1}{1+\cfrac{x}{x+1}} = \cfrac{1}{\cfrac{2x+1}{x+1}} = \frac{x+1}{2x+1}$$

Then the equation $\dfrac{x+1}{2x+1} = x$ becomes $x+1 = 2x^2 + x$ which is equivelent to $x^2 = \dfrac{1}{2}$. So $x_{1,2} = \pm\dfrac{\sqrt{2}}{2}$.

Problem 2: Using the double angle formula of cosine, we have

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha) = 1 - 2\left(\frac{1}{3}\right)^2 = \frac{7}{9}$$

Problem 3: We may asuume that $AP = 4a$ for some $a > 0$. Then $PB = 2a$, $DQ = QC = 3a$. Using the formula of the area of a trapezoid we obtain $\dfrac{Area(APQD)}{Area(PBCQ)} = \dfrac{(3a+4a)(6a)/2}{(3a+2a)(6a)/2} = \dfrac{7}{5}$
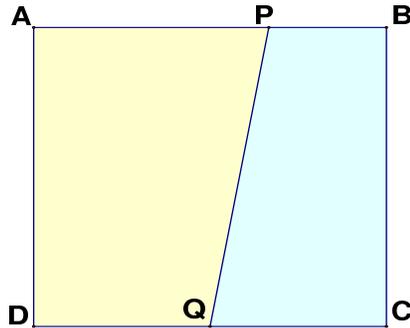
Problem 4: The probability of picking an even number or an odd number is $\frac{1}{2}$. We note that the product of two whole numbers is even as long as one of the numbers is even. Out of four possible outcomes

$$\{even, even\}, \{even, odd\}, \{odd, even\}, \{odd, odd\}$$

there are three even products. Therefore, the probability is $\frac{3}{4}$.

Problem 5: From the equation $x+y = xy$ we isolate x to obtain $x = \dfrac{y}{y-1}$ or $x = 1+\dfrac{1}{y-1}$. For any positive integer $y > 2$ or $y = 1$ the expression $\dfrac{1}{y-1}$ is not an integer. But when $y = 2$, we have $x = 2$. So $x + y = 4$.

Problem 6: The diagonal of a face is $10\sqrt{2}$ meters and then half of it is $5\sqrt{2}$. So, the Pythagorean theorem gives the length of the bee's path $\sqrt{10^2 + (5\sqrt{2})^2} = 5\sqrt{6}$ meters.
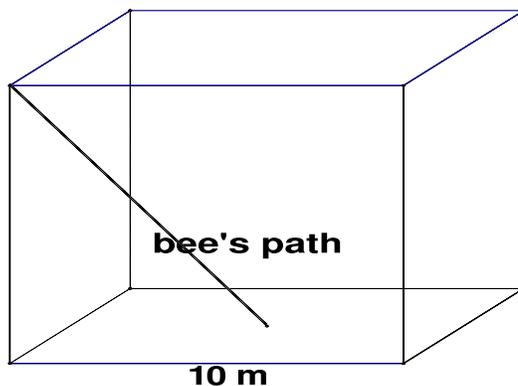
Problem 7: We let $y = 2^x$. Then the equation $2^{2x} + 2^x = 30$ becomes $y^2 + y = 30$, which has the solutions $y_1 = 5$ and $y_2 = -6$. Since $y = 2^x > 0$, the only possible solution is y_1. So $x = \log_2 5$.

Problem 8: Since $x^2 \geq 1$ and $y^2 \geq 1$ for nonzero integers x and y we see that $x^2 + y^2 \geq 2$ and the answer is $TRUE$.

Round II

Problem 1: Expanding the left side of the equation yields

$$1 + x + x^2 + x^3 = \frac{1}{x-1}.$$

Then multiplying each side by $x - 1$, we have

$$x^4 - 1 = 1 \quad \text{or} \quad x^4 = 2.$$

Therefore, $x_{1,2} = \pm\sqrt[4]{2}$ and $x_{3,4} = \pm\sqrt[4]{2}\,i$.

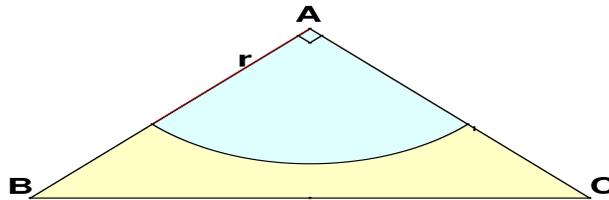**Problem 2:** Since $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, we have

$$n\left(\frac{\sqrt{3}}{2}\right) > 30.$$

Squaring the both sides of the inequality yields

$$n^2\left(\frac{3}{4}\right) > 900,$$

which is equivalent to the inequality $n^2 > 1200$. Note that $34^2 = 1156$ and $35^2 = 1225$. The answer is $n = 35$.

Problem 3: Let AB=a. Then the area of the triangle ABC is $a^2/2$. The area of the arc sector with radius r is $\pi r^2/4$. Then $\pi r^2/4 = a^2/4$. Hence $r = a/\sqrt{\pi}$.

Problem 4: A number n is divisible by 167 if and only if n is a integer multiple of 167. Note that $2004 = (12)(167)$. There are 12 integer multiples of 167 from 1 to 2004. Therefore the chance is $\dfrac{12}{2004} = \dfrac{1}{167}$.

Problem 5: Factoring the left side of the equation $x^2 - xy = 2$, we have $x(x - y) = 2$. Since x and y must be positive integers and the number 2 has only two positive factors 1 and 2, it is must be true that the factor $x = 2$ and the factor $x - y = 1$. Hence $x = 2$ and $y = 1$.

Problem 6: The height of the cone is 4 meters. hence the volume is $V = \frac{\pi \times 3^2 \times 4}{3} = 12\pi$.
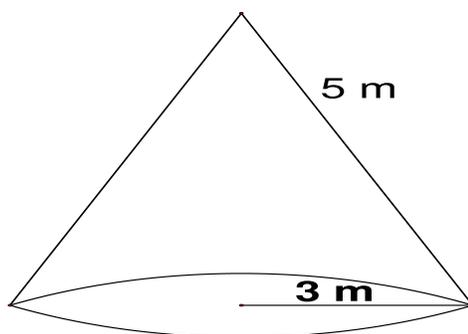
Problem 7: Using properties of logarithm, we write the equation

$$\ln x + \ln(x + 20) = \ln 3 + 2 \ln 10$$

as

$$\ln x(x + 20) = \ln 300.$$

So $x(x + 20) = 300$. The positive solution of this equation is $x = 10$.

Problem 8: Using binomial expansion, we note that the terms with odd power of $\sqrt{2}$ in the expansion are canceled out and that the terms with even power of $\sqrt{2}$ in the expansion are whole numbers. So the number M is a whole number and the answer is A.