45^{th} Ciphering Competition, March 2nd, 2019

Round 1

1. Find x for which

$$\frac{52}{\frac{11x-4}{13}-5}=13.$$

2. A box contains 7 balls, numbered 1, 2, 3, 4, 5, 6 and 7. If 4 balls are drawn simultaneously at random, what is the probability that the sum of the numbers drawn is even?

3. Compute

$$\sum_{k=1}^{100}\max(k,100-k).$$

4. What is the largest integer that is a divisor of

$$(n+1)(n+3)(n+5)(n+7)(n+9)$$

for all even positive integers n?

5. How many two digit numbers (in base 10) increase by 75% when their digits are reversed?

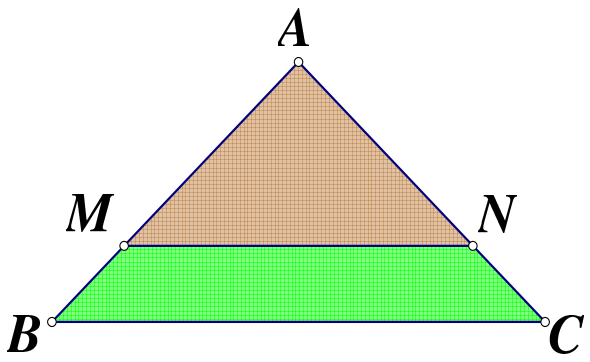
6. The equation $2019=a^2-b^2$ in a and b, for $a,b\in\{1,2,...,1000\}$, has a unique solution. Find this solution! (Write the answer as a pair (a,b).)

7. Four points are located on a line. When the distances between these points (measured with the same unit) are listed in non-decreasing order, we have

$$1,3,4,x,8 \text{ and } 9.$$

Find the value of x.

8. The line segment $\overline{MN}$ in the figure below is parallel to the side $\overline{BC}$ of an isosceles right triangle $\triangle ABC$ ($\overline{AB} = \overline{AC} = 1\ ft$). Find $\overline{MN}$ (in feet) knowing that $\overline{MN}$ divides the triangle ABC in two regions of equal area.



Round 2

1. The equation $x^2 = x+1$ has two real solutions: one is so called the Golden Ratio and denoted by ϕ, and the other say ψ. Find $\phi^3 + \psi^3$.

2. A box contains 7 balls, numbered 1, 2, 3, 4, 5, 6 and 7. If 4 balls are drawn simultaneously at random, what is the probability that the sum of the numbers drawn is odd?

3. Compute

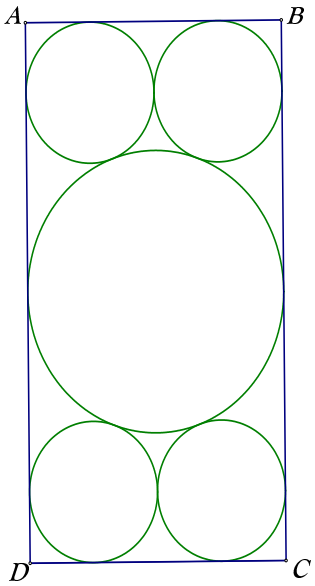
$$\sum_{k=1}^{100} \min(k, 100-k).$$

4. Five points are located on a line. When the distances between these points (measured with the same unit) are listed in non-decreasing order, we have

$$2, 4, 5, 7, 8, x, 13, 15, 17 \text{ and } 19.$$

Find the value of x.

5. The rectangle $ABCD$ shown below, contains five circles tangent to each other and to the sides of the rectangle. If $AB = 4$, what is the length of the side BC (measured with the same units)?



6. Let $f(x)$ denote the sum of the digits (in base 10) of the positive integer x. For instance, $f(5) = 5$ and $f(2019) = 2 + 0 + 1 + 9 = 12$. For how many two-digit values of x is $f(f(x)) = 3$?

7. The equation $2019 = a^2 - b^2$ in a and b, for $a, b \in \{1000, 1001, ...\}$, has a unique solution. Find this solution! (Write the answer as a pair (a, b).)

8. Find the sum of the digits of $10^{2019} - 2019$.

Backup problems:

1. Using the standard xy-coordinate plane, we consider the triangle whose vertices are given by the coordinates $(0, 0)$, $(1, 7)$, and $(5, 3)$. Calculate the area, in square units, of this triangle.

2. The circle of equation $x^2 + y^2 = 1$ intersects the line $y = 7x + 5$ at two points A and B. If the center of the circle is O, find the measure of the angle $\angle AOB$.

3. A square of perimeter 20 inches is inscribed in a disk as in the figure below. Find the area of the disk (in square inches).