

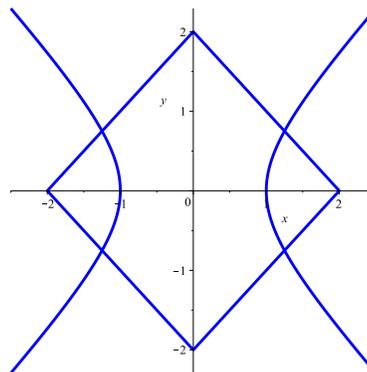
**Solutions for the 45th Annual Columbus State Invitational Mathematics
Tournament**

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1. How many solutions (x, y) , does the system of equations

$$\begin{cases} |x| + |y| = 2, \\ x^2 - y^2 = 1, \end{cases}$$

- have? (A) 1 (B) 2 (C) 3
(D) 4 (E) 5



Answer: We get 4 solutions by just plotting the two graphs (a square and a hyperbola). In fact the solutions are

$$(x, y) \in \left\{ \left(\frac{5}{4}, \frac{3}{4} \right), \left(\frac{5}{4}, -\frac{3}{4} \right), \left(-\frac{5}{4}, \frac{3}{4} \right), \left(-\frac{5}{4}, -\frac{3}{4} \right) \right\}.$$

2. What is the largest possible remainder that can be obtained when a two-digit number is divided by the sum of its digits?

- (A) 10 (B) 11 (C) 12 (D) 15 (E) 17

Answer: We have $79 = 16(4) + 15$. So the correct answer is either 15 or 17. To get 17 as a remainder we need to have the two digits 8 and 9 but $89 = 17 \cdot 5 + 4$ and $98 = 17 \cdot 5 + 13$. It remains that the correct answer is 15. One can actually check that 16 doesn't work either (although this is not one of our possible answers).

3. In how many ways can 20 dollars be changed into dimes and quarters, with at least one of each coin being used?

- (A) 35 (B) 37 (C) 39 (D) 41 (E) 43

Answer: We need to solve the Diophantine equation $2000 = 10x + 25y$ for $x, y > 0$. Clearly y must be even, i.e., $y = 2z$ for some $z \in \mathbb{N}$. The equation becomes $200 = x + 5z$. Since $1 \leq z < 40$ we have 39 solutions.

4. The number $\sqrt{24 + \sqrt{572}}$ can be written in the form $\sqrt{a} + \sqrt{b}$, for two positive integers a and b . What is $|a - b|$?

(A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

Answer: If we square $\sqrt{24 + \sqrt{572}} = \sqrt{a} + \sqrt{b}$ we get $a + b = 24$ and $ab = 572/4 = 143 = 11 \cdot 13$. Hence $\sqrt{24 + \sqrt{572}} = \sqrt{13} + \sqrt{11}$ and so B is the correct answer.

5. How many prime numbers divide $M := 5^6 - 1$?

(A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Answer: Using the formula $a^2 - b^2 = (a - b)(a + b)$ we see that M is divisible by $5^3 - 1 = 124 = 4 \cdot 31$ and $\frac{M}{124} = 126 = 2 \cdot 3^2 \cdot 7$. The answer is then D .

6. Find the remainder of the division $(x^4 - 4x^3 + 6x^2 - 3x) \div (x^2 - 2x + 1)$

(A) x (B) $\boxed{x - 1}$ (C) $x - 2$ (D) $x - 3$ (E) $x - 4$

Solution: Answer is B . We observe that $(x^4 - 4x^3 + 6x^2 - 3x) = (x - 1)^4 + x - 1$ and so the remainder must be $x - 1$.

7. A fair coin is flipped three times. What is the probability that at least one head will be thrown?

(A) 0.125 (B) 0.25 (C) 0.75 (D) 0.9 (E) $\boxed{0.875}$

Solution: Answer E . We compute the probability of the complementary event, i.e., ‘each time we flip a tail’. This probability is $\frac{1}{2^3} = 0.125$. Hence, the probability of the event required is $1 - \frac{1}{2^3} = 0.875$.

8. When expanded as a decimal, the fraction $\frac{1}{97}$ has a repetend (the repeating part of the decimal) that begins right after the decimal point and is 96 digits long:

$$\frac{1}{97} = 0.01030927835\dots x67\dots$$

If the last three digits of the repetend are $x67$, what is the digit x ?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: We need to have, $x67 \cdot 97 = 9700x + 6499 = \dots 999$ which determines x uniquely: $x = 5$.

9. What is the number of solutions in x (real values) of the equation

$$|x - 3| + |3x - 5| + |2x - 7| = 4?$$

- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: The answer is A since by triangle inequality, we have $|x - 3| + |3x - 5| + |2x - 7| \geq |x - 3 + 2x - 7 - (3x - 5)| = 5 > 4$.

10. Assuming that t is the only real solution of the cubic equation $x^3 = x + 2$, then there are unique integers m , n and p so that $t^8 = mt^2 + nt + p$. Find $m + n - p$.

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: If we square the given equation we get $x^6 = x^2 + 4x + 4$ and then

$$x^8 = x^4 + 4x^3 + 4x^2 = x^2 + 2x + 4(x + 2) + 4x^2 = 5x^2 + 6x + 8.$$

Hence, we get $m + n - p = 3$ which gives C as the answer.

11. The sum

$$2 \cdot [\cos^2 1^\circ + \cos^2 2^\circ + \dots + \cos^2 90^\circ]$$

equals a two-digit number ab . What is $b - a$?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: Using the double angle formula get

$$M = \sum_{k=1}^{90} (1 + \cos 2k^\circ) = 90 + \sum_{k=1}^{90} \cos 2k^\circ.$$

Using the identity $\cos \alpha = -\cos(180^\circ - \alpha)$ we obtain

$$M = 90 + \cos 90^\circ + \cos 180^\circ + \sum_{k=1}^{44} [\cos 2k^\circ + \cos(180^\circ - 2k^\circ)] = 89.$$

Hence, the answer is A.

12. The recurrent sequence $\{x_n\}$ satisfies the recurrence $x_{n+1} = 2x_n(1-x_n)$ for every $n \geq 1$ and $x_1 = 1/2019$. Knowing that $\{x_n\}$ is convergent to L , what is L^{-1} ?

- (A) 0 (B) 1 (C) $\boxed{2}$ (D) 3 (E) 4

Solution: Since x_{n+1} converges to L , we have the equation in L : $L = 2L(1 - L)$. We either get $L = 0$ or $L = \frac{1}{2}$. It is easy to see that x_n is non-decreasing and $x_n > 0$ for all n . Indeed, $x_{n+1} \leq [x_n + (1 - x_n)]^2/2 = 1/2$ and so $x_{n+1} \geq x_n$ is equivalent to $2(1 - x_n) \geq 1$ or $x_n \leq 1/2$. This implies $L^{-1} = 2$ and then the answer is \boxed{C} .

13. The following quadratic equations $15x^2 - 19x + 6 = 0$ and $21x^2 - 17x + 2 = 0$ have a common solution. What is the sum of the other solutions?

- (A) $\frac{24}{35}$ (B) $\frac{22}{35}$ (C) $\frac{28}{35}$ (D) $\frac{20}{35}$ (E) $\boxed{\frac{26}{35}}$

Solution: Answer is E. The common root can be found by eliminating x^2 from the two equations: $21(15x^2 - 19x + 6) - 15(21x^2 - 17x + 2) = 0$ or $-144x + 96 = 0$. This gives $x = \frac{2}{3}$. Then the other roots are $x_2 = \frac{3}{5}$ and $x_3 = \frac{1}{7}$. Hence $x_2 + x_3 = \frac{26}{35}$.

14. (2) The numbers a_1, a_2, \dots, a_6 are positive consecutive integers whose sum is 2019. What is the value of

$$\frac{6a_4 - a_1 - 2a_2 - 3a_3}{10}?$$

- (A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: Answer is A. One can check that $2019 = 334 + \dots + 339$, but the expression we have is independent of where we start with a_1 . Indeed, if $a_1 = a, a_2 = a + 1, \dots$, we get

$$\frac{6a_4 - a_1 - 2a_2 - 3a_3}{10} = \frac{6(a + 3) - a - 2(a + 1) - 3(a + 2)}{10} = 1.$$

15. The equation in x ,

$$(x-1)^5 + (x-2)^5 + (x-3)^5 + (x-4)^5 + (x-5)^5 = 0$$

has five solutions (including the pure complex ones). What is their product?

- (A) 875 (B) $\boxed{885}$ (C) 895 (D) 785 (E) 685

Solution: Answer is B . Using Viète's relations the product P of all roots is equal to $(-1)^5 \frac{c_5}{c_0}$ given that the equation is written as

$$c_0x^5 + c_1x^4 + \dots + c_5 = 0.$$

We can see that $c_0 = 5$ and $c_5 = -(1^5 + 2^5 + 3^5 + 4^5 + 5^5) = 4425$. Hence, $P = \frac{4425}{5} = 885$.

16. (2) If $\log_A B$ means the logarithm of B in base A , solve the equation $\log_9 x + \log_{x^2} 3 = 1$ for x .

- (A) 9 (B) 27 (C) 81 (D) $\boxed{3}$ (E) 1

Solution: Using the change of base formula and denoting $\log_3 x = t$ we have $\frac{t}{2} + \frac{1}{2t} = 1$. Solving for t we get $t = 1$ and so $x = 3$.

17. Find the sum of the number $N = 1234$ and all other four digit numbers obtained by permuting the digits of N in all possible ways.

- (A) 33330 (B) $\boxed{66660}$ (C) 999990 (D) 55550 (E) 11110

Solution: Answer B . Since there are $3! = 6$ permutations of three objects, 1 is the first digit 6 times in all these permutations. The same is true for each one of the digits of N . Hence, the sum is equal to $6 \cdot (1 + 2 + 3 + 4) \cdot 1111 = 66660$.

18. The cubic equation $x^3 - 12x^2 + 44x + m = 0$ has three solutions in arithmetic progression. Find m .

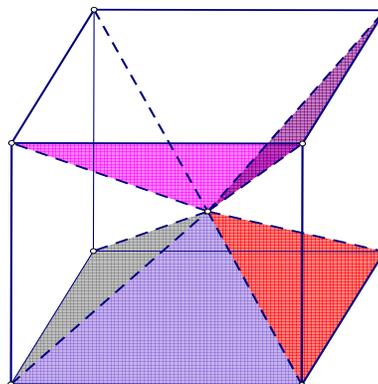
- (A) 50 (B) $\boxed{-48}$ (C) -55 (D) 20 (E) -35

Solution: Answer B . Assuming $x_1 = a - r$, $x_2 = a$ and $x_3 = a + r$, using Viète's relations $x_1 + x_2 + x_3 = 3a = 12$. This means $a = 4$. Also, $x_1x_2 + x_2x_3 + x_3x_1 = 44$ which implies

$$a^2 - r^2 + 2a^2 = 44 \implies r^2 = 4 \implies r = \pm 2.$$

This means $m = -2 \cdot 4 \cdot 6 = -48$.

19. A cube is dissected into six pyramids by connecting a given point in the interior of the cube with each vertex of the cube. The volumes of five of these pyramids are 2, 5, 10, 11 and 14. What is the volume of the sixth pyramid?



- (A) 4 (B) 5 (C) 6
 (D) 7 (E) 8

Answer: The pyramids can be grouped in pairs which have the sum of their volumes the same value (a third of the volume of the cube). Pyramids in a pair have bases which are opposite faces of the cube. Since $2 + 14 = 5 + 11 = 10 + 6$ we see that the answer must be 6.

20. The positive integer N has exactly six distinct (positive) divisors including 1 and N . The product of five of these divisors is 648. What is the value of the sixth missing divisor?

- (A) 18 (B) 9 (C) 6 (D) 2 (E) 1

Answer: We begin with the factorization $648 = 2^3(3^4)$. Hence the only primes in the factorization of N are 2 and 3. The power of 2 cannot be more than 2. So one gets $N = 18 = 2(3^2) = q \cdot p^2$, where $p = 3$ and $q = 2$. Since the product of the divisors is

$$P = 1 \cdot p \cdot p^2 \cdot q \cdot qp \cdot qp^2 = p^6 q^3$$

we see that $p^2 = 9$ is missing, i.e., $P = 648(9)$. Hence, the answer is B.

21. Suppose that a and b are positive integers such that $2019 > a > b > 1$, and the expression

$$E := \frac{a+b}{a-b} + \frac{a-b}{a+b}$$

attains its smallest possible value. What is the units digit (in base 10) of $a + 3b$?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: If we denote $\frac{a}{b}$ by t ($t > 1$), we can write

$$\frac{a+b}{a-b} + \frac{a-b}{a+b} = \frac{t+1}{t-1} + \frac{t-1}{t+1} = s + \frac{1}{s},$$

where $s = \frac{t+1}{t-1} = 1 + \frac{2}{t-1} > 1$. As a function of s , the expression above has derivative $1 - \frac{1}{s^2} = \frac{s^2-1}{s^2} > 0$, which means the function is strictly increasing on $(1, \infty)$. Therefore, E 's minimum is where s is at its minimum. This implies that t must be at its maximum: $t = \frac{2018}{2} = 1009$. Then $a + 3b = 2018 + 3(2) = 2024$ and so D is the correct answer.

22. A natural number is called a perfect power if it is equal to a^b for some a and b natural numbers. The list of perfect powers begins with 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, etc. The sum S_n of the first n -perfect powers gives a sequence $S_1 = 1$, $S_2 = 1 + 4 = 5$, $S_3 = 1 + 4 + 8 = 13$, and so on. Knowing that $S_n = 2019$ for some n , find the sum of the digits of n .

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: One needs to add a few more perfect numbers to the given list: 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196, 216, 225, 243 and add them up. This gives $S_{22} = 2019$. So, the answer is D .

23. A box has 5 red balls and 3 blue balls. Bill picks one ball at the time, from the box, without replacement, until all of balls of the same color are out. If $\frac{p}{q}$ is the probability that the last ball Bill picks is blue, written as a reduced fraction, what is $q - p$?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: We can think of this experiment as choosing 7 balls all at once. One ball will be left. If what is left is a red ball, this is then a favorable event since we had picked all of the blue ones within the 7 ones. If what is left is a blue ball, we had picked all the red ones first, and so this is an unfavorable event. We obtain $\binom{7}{3} = \frac{7(6)(5)}{1(2)(3)} = 35$ possibilities for the first case and $\binom{7}{5} = 21$ arrangements for the second case. Therefore, the probability is $\frac{35}{35+21} = \frac{5}{8}$. So, the answer is C .

24. The real valued function f defined by

$$f(x) = \sum_{j=0}^{2019} |x - j|, x \in \mathbb{R},$$

has a minimum value of M^2 . What is the sum of the digits of M ?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: Using triangle inequality, we can write

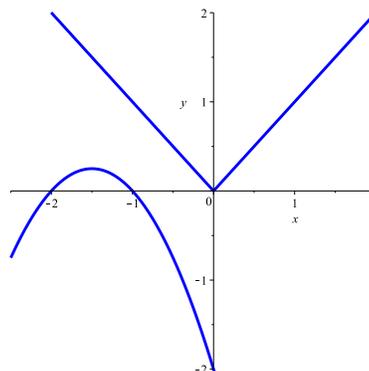
$$f(x) = \sum_{j=0}^{1009} (|x-j| + |x-(2019-j)|) \geq \sum_{j=0}^{1009} |2019-2j| = 2019(1010) - 1009(1010) = 1010^2.$$

This is indeed the minimum since $f(1009.5) = 1010^2$. Therefore, $M = 1010$ and the answer is *B*.

25. For every real number x , a vertical segment connecting the graphs of

$$y = |x| \text{ and } y = -x^2 - 3x - 2$$

can be drawn (segment \overline{AB} with $A = (x, |x|)$ and $B = (x, -x^2 - 3x - 2)$). Find the smallest possible length of these vertical segments.



- (A) 1 (B) 2 (C) 3
(D) 4 (E) 5

Answer: We need to minimize the continuous function

$$g(x) = |x| + x^2 + 3x + 2 = \begin{cases} x^2 + 2x + 2 & \text{if } x < 0 \\ x^2 + 4x + 2 & \text{if } x \geq 0. \end{cases}$$

This function is continuous and piecewise differentiable (except $x = 0$). We have

$$g'(x) = \begin{cases} 2(x+1) & \text{if } x < 0 \\ 2(x+2) & \text{if } x > 0. \end{cases}$$

This shows that g has a global minimum at $x = -1$, equal to $g(-1) = 1$, and so *A* is the correct answer.

26. The Golden ratio ϕ is the famous number which equals $\frac{1+\sqrt{5}}{2}$. Consider the real valued function f defined for all $z \neq \phi$, by $f(z) = \frac{1}{\phi-z}$. The composition of f with itself, n -times is denoted by $f^{[n]}$. For example, $f^{[2]} = f \circ f$, $f^{[3]} = f \circ f \circ f$, etc. Calculate $f^{[2019]}(2)$.

- (A) $3 - 2\phi$ (B) 2 (C) $\frac{2\phi - 1}{2}$ (D) $\frac{\phi+2}{5}$ (E) $-\phi - 1$

Answer: It is easy to check that $\phi^2 = \phi + 1$. First we observe that $f(2) = \frac{1}{\phi-2} = -\phi - 1$

since $(\phi - 2)(\phi + 1) = \phi^2 - \phi - 2 = -1$. Then, we continue and find

$$f^{[2]}(2) = f(f(2)) = \frac{1}{\phi + \phi + 1} = \frac{1}{2\phi + 1}, f^{[3]}(2) = \frac{2\phi + 1}{2\phi^2 + \phi - 1} = \frac{2\phi + 1}{3\phi + 1}$$

With this, we get

$$f^{[4]}(2) = \frac{3\phi + 1}{3\phi^2 + \phi - 2\phi - 1} = \frac{3\phi + 1}{2\phi + 2} = \frac{(3\phi + 1)(2 - \phi)}{-2} = \frac{6\phi + 2 - 3\phi^2 - \phi}{2} = \frac{2\phi - 1}{2}.$$

Finally, $f^{[5]}(2) = 2$ and we see that the sequence repeats with a period of 5. Hence $f^{[2019]}(2) = f^{[4]}(2) = \frac{2\phi - 1}{2}$ giving C as the correct answer.

27. Assume that a is a positive real number. A recurrent sequence $\{x_n\}$ is defined by $x_1 = 1$ and

$$x_{n+1} = \frac{x_n}{1 + ax_n}$$

for all natural numbers $n \geq 1$. Knowing that

$$x_{674} = \frac{1}{2020}$$

find a .

- (A) 2 (B) $\boxed{3}$ (C) 4 (D) 5 (E) 6

Answer: We observe that $\frac{1}{x_{n+1}} - \frac{1}{x_n} = a$. This implies that $\frac{1}{x_{674}} - \frac{1}{x_1} = 673a$ or $a = \frac{2019}{673} = 3$. Therefore, the answer is B .

28. We have a function f defined for all real numbers x by the functional equation

$$f(x) + xf(-x) = 1.$$

Knowing that the positive integer M which is equal to

$$2019 \cdot f\left(\frac{12}{23}\right)$$

is a 3-digit number (in base 10), what is its first digit (the hundreds place)?

- (A) 4 (B) 5 (C) 6 (D) $\boxed{7}$ (E) 8

Answer: We need to solve the system

$$\begin{cases} f(x) + xf(-x) = 1, \\ f(-x) - xf(x) = 1. \end{cases}$$

It is easy to see that this leads to $f(x) = \frac{1-x}{1+x^2}$ and so

$$2019 \cdot f\left(\frac{12}{23}\right) = 3 \cdot 11 \cdot 23 = 759,$$

which shows that D is the correct answer.

29. Find the number of positive integer solutions (x, y) , of the following Diophantine equation:

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$$

- (A) 4 (B) 5 (C) 6 (D) $\boxed{7}$ (E) 8

Answer: The equation can be written as $(y + 12)(12 - x) = 144$. Since $144 = 12^2 = 3^2 \cdot 2^4$ has $3 \cdot 5 = 15$ positive integer divisors, we have $(15 - 1)/2 = 7$ such pairing of the divisors $(144, 1)$, $(72, 2)$, $(48, 3)$, $(36, 4)$, $(24, 6)$, $(18, 8)$, and $(16, 9)$. Each pair gives a solution for (x, y) . For instance, $(72, 2)$ leads to $y = 60$ and $x = 10$. Hence, the answer is D .

30. The expression in terms of the real valued variable x ,

$$E := \sec^2(x) + 2 \csc^2(x) + 3 \sec^2(x) \csc^2(x)$$

has the minimum value equal to $m + 4\sqrt{n}$ over the interval $(0, \frac{\pi}{2})$, for some natural numbers m and n . What is $m - n$?

- (A) 1 (B) 2 (C) 3 (D) $\boxed{4}$ (E) 5

Answer: If we denote $t = \tan(x)$, then $E = 1 + t^2 + 2(1 + \frac{1}{t^2}) + 3(1 + t^2)(1 + \frac{1}{t^2})$. This can be simplified to $E = 9 + 4t^2 + \frac{5}{t^2} \geq 9 + 4\sqrt{5}$. Therefore, the correct answer is D .

31. Assume that x is not equal to zero. If we eliminate α from the two equalities

$$\begin{cases} x^2 + y \cos^2 \alpha = x \sin \alpha \cos \alpha \\ x \cos 2\alpha + y \sin 2\alpha = 0, \end{cases}$$

we get $4x^2 + 4y = n$. Find n .

- (A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Answer: Using the double angle formula we have

$$\begin{cases} x \sin 2\alpha - y \cos 2\alpha = 2x^2 + y, \\ x \cos 2\alpha + y \sin 2\alpha = 0. \end{cases}$$

We obtain

$$\begin{cases} x = (2x^2 + y) \sin(2\alpha), \\ y = -(2x^2 + y) \cos(2\alpha). \end{cases}$$

Hence $x^2 + y^2 = (2x^2 + y)^2$ or $4x^4 + 4x^2y = x^2$. Simplifying by x^2 we see that $4x^2 + 4y = 1$ (Answer A).

32. A number t is randomly chosen (uniform distribution) from the unit interval $[0, 1]$. Calculate the expectation of the random variable $X(t) = \max\{t, 1 - t\}$.

- (A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{4}{5}$ (D) $\frac{3}{5}$ (E) $\boxed{\frac{3}{4}}$

Solution: The expectation is equal to $E(X) = \int_0^{1/2} (1 - t) dt + \int_{1/2}^1 t dt = \frac{3}{4}$, so the answer is E.

33. The equation $9x^3 = a + \ln x$ has a unique positive solution in x given this unique particular positive real number a . What is the value of $e^{a-1/3}$?

- (A) 7 (B) 6 (C) 5 (D) 4 (E) $\boxed{3}$

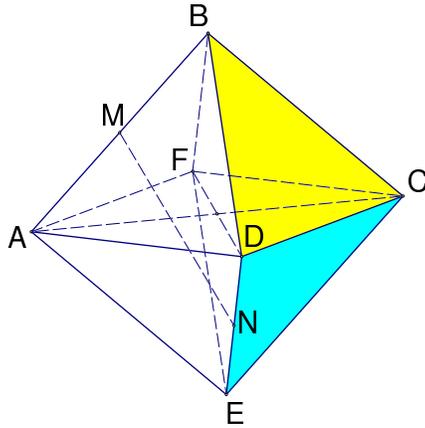
Solution: Answer is E. The function $f(x) = 9x^3 - \ln x$ has a derivative equal to $f'(x) = \frac{27x^3 - 1}{x}$ which shows that f has a global minimum at $x = \frac{1}{3}$ of $f(1/3) = 1/3 - \ln(1/3) = 1/3 + \ln 3$. Clearly a must be this minimum, and so $e^{a-1/3} = e^{\ln 3} = 3$.

34. The graph of the function g is the reflection of the graph of $f(x) = 2x^3 - 3x^2 + 3x$ defined for all real x , across the line $y = x$. What is $g'(1)$ (i.e. $\frac{dg}{dx}(1)$) ?

- (A) $\frac{3}{2}$ (B) $\frac{1}{3}$ (C) $\frac{3}{4}$ (D) $\frac{2}{5}$ (E) $\boxed{\frac{2}{3}}$

Solution: Answer is E. One reads from the given information that g is the inverse of f . So, $g(f(x)) = x$ implies $g'(f(x))f'(x) = 1$. Then for $x = 1/2$ we get $g'(f(1/2))f'(1/2) = 1$ or $g'(1) = \frac{2}{3}$.

35. In the accompanying figure we have a regular octahedron $ABCDEF$ of side-lengths equal to 4. The point M is the midpoint of the side \overline{AB} and the point N is the midpoint of the side \overline{DE} . How long is the segment \overline{MN} (same units)?



- (A) $2\sqrt{5}$ (B) 4 (C) $2\sqrt{3}$
 (D) $2\sqrt{6}$ (E) $2\sqrt{2}$

Solution: Answer is C. We project the points M and N on the plane of the square $ADCF$, and notice they fall into the midpoints of \overline{AO} and \overline{OD} (where O is the center of the octahedron). Then using the Pythagorean theorem we get $MN^2 = 2^2 + (2\sqrt{2})^2 = 12$ and so $MN = 2\sqrt{3}$.

36. Find the limit of

$$L := \lim_{x \rightarrow 0} \frac{\sqrt[3]{5x+8} - 2}{x}$$

- (A) $5/12$ (B) $7/12$ (C) $1/12$ (D) $5/4$ (E) $5/3$

Solution: Answer is A. Using L'Hospital's Rule $L = \lim_{x \rightarrow 0} \frac{5}{3\sqrt[3]{(5x+8)^2}} = \frac{5}{12}$

37. Knowing that every solution of the equation $(\cos x)^2 - (\cos x) - 1 = 0$ is a solution of the equation $a(\cos 2x)^2 + b(\cos 2x) - 1 = 0$, what is $a + b$?

- (A) -1 (B) -2 (C) -3 (D) -4 (E) -5

Solution: Answer C. Using the double angle formula and denoting $t = \cos x$ we see that the second equation must have the solution $\cos(2x) = 2(\cos x)^2 - 1 = 2t^2 - 1 = 2t + 1$. This gives a new quadratic equation $a(2t + 1)^2 + b(2t + 1) - 1 = 0$, which must have coefficients proportional to the quadratic equation $t^2 - t - 1 = 0$. One can easily solve the system and get $a = 1$ and $b = -4$.

38. How many integer ordered pairs (x, y) of solutions does the Diophantine equation $x^2 + y^2 = x^3 + y^3$ have ?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: If we denote $x + y = s$ and $xy = p$ we have $s^2 - 2p = s(s^2 - 3p)$. Solving for p we obtain

$$p = \frac{s^3 - s^2}{3s - 2} \implies 27p = 9s^3 - 3s - 2 - \frac{4}{3s - 2}.$$

Then we see that $3s - 2$ must be ± 1 , ± 2 or ± 4 . Then $3s$ is in the set $\{-2, 0, 1, 3, 4, 6\}$. This gives $s \in \{0, 1, 2\}$. In the first case, $s = 0$, we get $p = 0$ which corresponds to $x = y = 0$. For $s = 1$ we get $p = 0$ which corresponds to $x = 0$ and $y = 1$ or $x = 1$ and $y = 0$. Finally, if $s = 2$ we have $p = 1$ and which implies $x = y = 1$. So the answer is D .

39. For positive real numbers a and b , the equation $x + \ln(ab) = \ln(x + a) + \ln(x + b)$ has a unique real solution in x . What is the value of $1/a + 1/b$?

(A) $\boxed{1}$ (B) 2 (C) 3 (D) 4 (E) 5

Solution: The function $f(x) = x + \ln(ab) - (\ln(x + a) + \ln(x + b))$ has a derivative equal to $f'(x) = 1 - \frac{1}{x+a} - \frac{1}{x+b}$. Since $f(0) = 0$, then $f'(0)$ must be equal to 0 otherwise $f(x) = 0$ has more than one solution. Hence, the answer is A .

40. Two points are chosen at random on a circle of radius 1. What is the expectation of the distance between them ?

(A) $2/3$ (B) $\frac{\pi}{3}$ (C) $\frac{3}{\pi}$ (D) $\boxed{\frac{4}{\pi}}$ (E) $\frac{3}{2}$

Solution: Answer D . We may assume that one of the points is fixed, say $(1, 0)$ on the unit circle $x^2 + y^2 = 1$ parameterized by $x = \cos t$ and $y = \sin t$ with $t \in [0, 2\pi]$. We have

$$E(X) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(\cos t - 1)^2 + \sin^2 t} dt = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2(1 - \cos t)} dt \implies$$

$$E(X) = \frac{1}{2\pi} \int_0^{2\pi} 2 \sin \frac{t}{2} dt = \frac{1}{\pi} (-2 \cos \frac{t}{2}) \Big|_0^{2\pi} = \frac{4}{\pi}.$$

41. It is known that every positive integer can be written as a sum of non-consecutive Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 2$) in a unique way. Taking into consideration this writing for 2019,

$$2019 = F_{n_1} + F_{n_2} + \cdots + F_{n_k},$$

find k .

(A) 1 (B) 2 (C) 3 (D) 4 (E) $\boxed{5}$

Solution: Answer *E*. One can check that

$$2019 = F_{17} + F_{14} + F_9 + F_6 + F_4.$$

42. How many positive integers n have the property that both $\sqrt{2n - 4030}$ and $\sqrt{2n + 4030}$ are integers?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: The system $2n + 4030 = x^2$ and $2n - 4030 = y^2$ implies that

$$x^2 - y^2 = (x + y)(x - y) = 2 \cdot 4030 = 4 \cdot 5 \cdot 13 \cdot 31.$$

Hence $(x+y)$ and $(x-y)$ must be both even. Then $d_1 = (x+y)/2$ and $d_2 = (x-y)/2$ are odd divisors of $1 \cdot 5 \cdot 13 \cdot 31 = d_1 d_2$, $d_1 > d_2$, with the sum $d_1 + d_2 = x = \sqrt{2n + 4030} > \sqrt{4030}$ ($x \geq 65$). Since $d_1 > (65 + 1)/2 = 32$, we have $d_1 \in \{65, 155, 403, 2015\}$. The set of the values of n is then $\{2593, 12097, 81217, 2030113\}$ which gives *D* the correct answer.

43. Find the smallest natural number n such that there exists a choice of signs that makes the following equality true

$$\pm 1 \pm 2 \pm 3 \cdots \pm n = 2019.$$

- (A) 61 (B) 62 (C) 63 (D) 64 (E) 65

Solution: Answer *E*. Since $\sum_{k=1}^{63} k = \frac{63 \cdot 64}{2} = 2016 < 2019$, $n = 63$ is not going to cut it. For $n = 64$ the corresponding sum, independent of the choice of signs, it is going to be an even number and so it cannot be 2019. For $n = 65$ we have

$$2019 = 1 + 2 - 3 + 4 - 5 + 6 - 7 + 8 - 9 + 10 - 11 + 12 - 13 + 14 - 15 + 16 + \sum_{k=17}^{65} k.$$

44. Find the closest integer to the sum

$$\sum_{k=1}^{2019} \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}}.$$

- (A) 2019 (B) 2020 (C) 2021 (D) 2022 (E) 2023

Answer: With a little algebra one obtains that $\sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}} = 1 + \frac{1}{k} - \frac{1}{k+1}$. We get then a telescoping sum which is equal to

$$\sum_{k=1}^{2019} \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}} = 2019 + 1 - \frac{1}{2020},$$

so the answer is *B*.

45. *The value of the integral*

$$I := \int_0^1 \frac{\sqrt{x} + \sqrt{1-x}}{1 + \sqrt{2x}} \sin \pi x dx$$

is equal to $\frac{\sqrt{n}}{m\pi}$ for some $m, n \in \mathbb{N}$. Find $3n - m$.

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: We substitute $x = 1 - t$ and get that

$$I = \int_1^0 \frac{\sqrt{1-t} + \sqrt{t}}{1 + \sqrt{2(1-t)}} \sin(\pi - \pi t)(-dt) = \int_0^1 \frac{\sqrt{1-t} + \sqrt{t}}{1 + \sqrt{2(1-t)}} \sin(\pi t) dt.$$

We then see that

$$2I = \int_0^1 (\sqrt{x} + \sqrt{1-x}) \left(\frac{1}{1 + \sqrt{2x}} + \frac{1}{1 + \sqrt{2(1-x)}} \right) \sin \pi x dx \implies$$

$$2I = \int_0^1 (\sqrt{x} + \sqrt{1-x}) \left(\frac{2 + \sqrt{2x} + \sqrt{2(1-x)}}{(1 + \sqrt{2x})(1 + \sqrt{2(1-x)})} \right) \sin \pi x dx \implies$$

$$2I = \sqrt{2} \int_0^1 \left(\frac{\sqrt{2x} + \sqrt{2(1-x)} + (\sqrt{x} + \sqrt{(1-x)})^2}{1 + \sqrt{2x} + \sqrt{2(1-x)} + 2\sqrt{x(1-x)}} \right) \sin \pi x dx.$$

This means $2I = \sqrt{2} \int_0^1 \sin \pi x dx = \sqrt{2} \left(-\frac{\cos \pi x}{\pi} \Big|_0^1 \right) = 2 \frac{\sqrt{2}}{\pi}$ which gives *E* as the answer.

46. *The positive integers x, y, z satisfy $x < y < z$ and*

$$\frac{1}{x} - \frac{1}{xy} - \frac{1}{xyz} = \frac{19}{97}.$$

Find the last digit of $x - y + z$ (in base 10).

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: The equation can be written as $97(yz - z - 1) = 19xyz$. Since $\gcd(z, yz - z - 1) = 1$ we conclude that z must divide 97 which happens to be a prime number. We have $z = 97$ because $z > 1$. Hence the equation becomes $97y - 98 = 19xy$. So, $y > 1$ must divide 98. From the divisors of 98, only $y = 49$ makes $97y - 98$ divisible by 19. Then $x = (97 \cdot 49 - 98)/(19 \cdot 49) = 5$. Therefore, $x - y + z = 5 - 49 + 97 = 53$ which means the answer is *C*.

47. The Taylor expansion of $f(x) = \frac{(x+1)(3-2x)}{(1-x)^3}$ around $x = 0$ has as one of its terms equal to $2019x^k$ for some positive integer k . The number k is a two digit number if written in base 10, i.e. $k = 10a + b$ with $a, b \in \{0, 1, \dots, 9\}$. What is $a - b$?

- (A) 1 (B) $\boxed{2}$ (C) 3 (D) 4 (E) 5

Answer: It is well known that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for all $|x| < 1$. By differentiation, one obtains that $\sum_{k=0}^{\infty} (k+1)x^k = \frac{1}{(1-x)^2}$ and $\sum_{k=0}^{\infty} (k+1)(k+2)x^k = \frac{2}{(1-x)^3}$. Since

$$f(x) = \frac{(x+1)(3-2x)}{(1-x)^3} = \frac{[2 - (1-x)][1 + 2(1-x)]}{(1-x)^3} = \frac{2}{(1-x)^3} + \frac{3}{(1-x)^2} - \frac{2}{(1-x)}$$

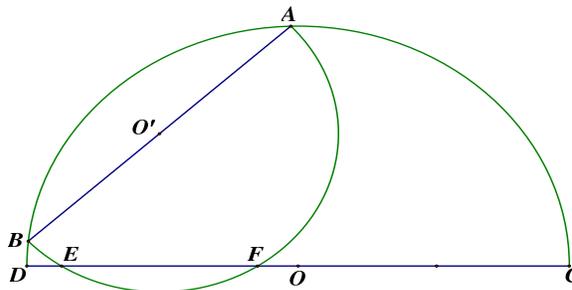
From these, the Taylor expansion of $f(x)$ can be computed:

$$f(x) = \sum_{k=0}^{\infty} [(k+1)(k+2) + 3(k+1) - 2]x^k = \sum_{k=0}^{\infty} [(k+3)^2 - 6]x^k.$$

Hence $2019 = (k+3)^2 - 6$ which implies $(k+3)^2 = 2025 = 45^2$. Hence $k = 42$ and the answer is *B*.

48. (5) In the adjacent figure, we have two semi-circles with centers O and O' . Knowing that $DE = 3$, $EF = 17$ and $FC = 27$, what is the sum of the digits of AB ?

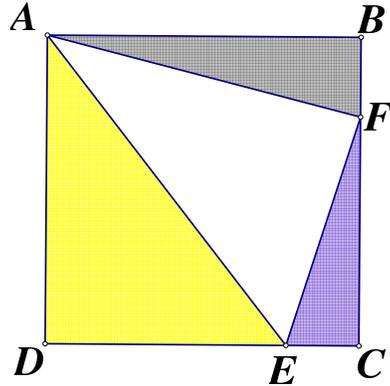
- (A) 1 (B) 2 (C) 3
(D) $\boxed{4}$ (E) 5



Answer: The formula for AB is given by $AB = \sqrt{n^2 + mn + pn + 2mp}$ where $m = DE$, $n = EF$ and $p = FC$. This can be derived using the power of a point with respect to a circle and the formula for the median. In this particular case we obtain $AB = 31$, and hence *D* is the correct answer.

49. In the adjacent figure $ABCD$ is a square and points E and F are on its sides \overline{DC} and \overline{BC} respectively. By $[XYZ]$ we denote the area of the triangle XYZ . Knowing that $[ADE] = 21$, $[ECF] = 2$ and $[ABF] = 18$, find $[AEF]$.

- (A) 11 (B) 12 (C) 13
 (D) 14 (E) 15



Answer: In general, if $AB = x$, $[ADE] = m$, $[ECF] = n$ and $[ABF] = p$ we get

$$x^2 = m + n + p + \sqrt{(m + n + p)^2 - 4mp}. \quad (1)$$

To derive this, we see that $[ADE] = x(DE)/2$, and so $DE = \frac{2m}{x}$. Then $[EFC] = (EC)(CF)/2$ which means

$$FC = \frac{2n}{EC} = \frac{2n}{x - \frac{2m}{x}} = \frac{2nx}{x^2 - 2m}.$$

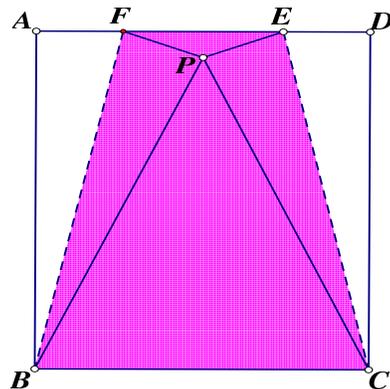
Finally, we have

$$[ABF] = (FB)x/2 = (x - FC)x/2 \implies 2p = x\left(x - \frac{2nx}{x^2 - 2m}\right),$$

which is a biquadratic equation in x . Solving this equation for x^2 , one obtains (1). From here, $[AEF] = x^2 - m - n - p$. In particular, given the information we have $[AEF] = \sqrt{(21 + 18 + 2)^2 - 4(21)(18)} = \sqrt{169} = 13$ and so $q = 13$. This gives answer C.

50. A rectangular piece of paper $ABCD$ with side lengths $AB = 34$ cm and $AD = 32$ is folded along the dashed line segments \overline{FB} and \overline{EC} , such that the points A and D arrive at the same point P . Find the area of the trapezoid $EFBC$.

- (A) 799 (B) 800 (C) 801
 (D) 802 (E) 803



Answer: Let us show in general that, if $AB = m^2 + n^2$, $AD = 2(m^2 - n^2)$ where

$m > n$, then

$$[EFBC] = \frac{(m-n)(m^2+n^2)(m^2+4mn+n^2)}{m+n}. \quad (2)$$

Indeed, triangles PFB and FAB are congruent. Similarly triangles PEC and EDC are congruent. Hence, triangle PBC is isosceles with height (corresponding to the base BC equal to $\sqrt{PB^2 - (BC/2)^2} = \sqrt{(m^2+n^2)^2 - (m^2-n^2)^2} = \sqrt{4m^2n^2} = 2mn$. The triangle PEF is also isosceles, and has a height equal to $AB - 2mn = (m-n)^2$. Since $PE = ED$ we have $(m-n)^2 = \sqrt{PE^2 - [(AD - 2PE)/2]^2}$. Solving for PE we get $PE = ED = \frac{(m-n)^4 + (m^2-n^2)^2}{2(m^2-n^2)}$ or

$$ED = \frac{2m^4 - 4m^3n + 4m^2n^2 - 4mn^3 + 2n^2}{2(m^2 - n^2)} = \frac{2(m^2 + n^2)(m - n)^2}{2(m^2 - n^2)}.$$

Then, $FE = AD - 2ED = 2(m^2 - n^2) - 2\frac{(m^2+n^2)(m-n)}{(m+n)} = 2(m-n)\frac{(m+n)^2 - (m^2+n^2)}{m+n}$ or $FE = \frac{4(m-n)mn}{m+n}$. This implies that $[EFBC] = (BC+EF)AB/2 = \frac{(m-n)(m^2+n^2)(m^2+4mn+n^2)}{m+n}$ which is (2).

In our case, $m = 5$ and $n = 3$ and so $[EFBC] = 799$, giving the answer A for this problem.