Ciphering 2020

Round 1.

1) Josh has a die with the numbers 1, 2, 3, 4, 5 and 6 on its six faces. Makayla has a die which is special: it has the numbers 2, 2, 3, 5, 5 and 5 on its six faces. When Josh and Makayla roll their dice the one with the larger number wins. If the two numbers are equal it is a draw. What is the probability that Makayla wins?
Answer: 4/9

2) What is the smallest positive integer n such that $\dfrac{n}{n+101}$ is equal to a terminating decimal?
Answer: 24

3) Let
$$f(x) = \frac{cx}{2x+3}.$$
Find the constant c so that $f(x) = f^{-1}(x)$, for all real numbers $x \neq -3/2$.
Answer: -3

4) Let n be a positive integer whose only digits are 0s and 1s. If $x = n \div 12$ and x is an integer, what is the smallest possible value of x?
Answer: 925

5) How many triangles ABC with $\angle CAB = 90^\circ$ and $AB = 20$ exist such that all sides have integer lengths?
Answer: 4

6) As n ranges over the positive integers, what is the maximum possible value for the greatest common divisor of $5n + 4$ and $7n + 2$?
Answer: 18

7) The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.
Answer: 15

8) There are 2020 marbles in a cane. The marbles are numbered from 1 to 2020. Marbles with equal digit sums have the same color and marbles with different digit sums have different colors. How many different colors of marbles are there in the cane?
Answer: 28

Round 2.

1) If the least common multiple of two 6-digit integers has 10 digits, then their greatest common divisor has at most how many digits?
Answer: 3

2) Let

$$f(n) = \begin{cases} n^2+1 & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

For how many integers n from 1 to 2020, inclusive, does

$$f(f(\cdots f(n)\cdots)) = 1,$$

for some number of applications of f?
Answer: 11

3) How many square units are in the area of the triangle whose vertices are the x and y intercepts of the curve $y = (x-3)^2(x+2)$?
Answer: 45

4) For $1 \leq n \leq 100$, how many integers are there such that $\dfrac{n}{n+1}$ is a repeating decimal?
Answer: 86

5) Let

$$f(x) = \frac{x}{\sqrt{1+x^2}}.$$

A sequence of functions is defined by $f^{(1)}(x) = f(x)$ and

$$f^{(n)}(x) = f(f^{(n-1)}(x)),$$

for $n \geq 2$. Find $f^{(99)}(1)$.
Answer: 1/10

6) What is the greatest common divisor of $2^{63}-1$ and $2^{54}-1$?
Answer: 511

7) The equations $x^2+ax+b=0$ and $x^2+bx+a=0$ both have real roots. If the sum of squares of the roots of the first equation is equal to the sum of squares of the roots of the second one, what is the value of $a+b$?
Answer: -2

8) Let z be a complex number such that $z^5 = 1$ and $z \neq 1$. Compute

$$z + \frac{1}{z} + z^2 + \frac{1}{z^2}.$$

Answer: -1

Backup Problems

1) If $x^2 + 5x + 3 = 0$, then what is the value of $x + \dfrac{3}{x}$?
Answer: -5

2) If a and b are the solutions of the equation $x^2 + 2019x + 2020 = 0$, what is the value of $\frac{1}{a} + \frac{1}{b}$?
Answer: $-2019/2020$.

3) If $\sin x + \cos x = \dfrac{1}{2}$, what is the value of $\sin(2x)$?
Answer: $-3/4$.