

Forty-Sixth Annual Columbus State Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
March 7rd, 2020

The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one “*best*” answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, -3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 4, 8, 10, 12, 14, 17, 21, 25, 27, 29, 33, 35, 37, 38, 39, 41, 42, 44, 45, 46.

Throughout the exam, \overline{AB} will denote the line segment from point A to point B and AB will denote the length of \overline{AB} . Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet.**

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

All numbers when referred to their digits are considered written in base 10 unless otherwise stated. Important facts and notations to consider:

- $2020 = 2^2 \cdot 5 \cdot 101$ (prime factorization)
- $[XYZW\dots]$ means the area of the polygon $XYZW\dots$

1. How many positive divisors does 2020 have?

- (A) 9 (B) 10 (C) 11 (D) 12 (E) 13

2. For what values of the real number a does the equation

$$x^2 - ax - a = 0$$

have two different complex zeros, which are not real numbers?

- (A) $a > 0$ (B) $a < 0$ (C) $0 < a < 4$
(D) $-4 < a < 0$ (E) $a < 4$

3. Let x and y be positive integers such that

$$\frac{2020!}{5^x \cdot 11^y}$$

is an integer. What is the largest possible value of $x + y$?

- (A) 695 (B) 697 (C) 699 (D) 701 (E) 703

4. * If $i = \sqrt{-1}$, find the sum

$$1 + i + i^2 + i^3 + \dots + i^{2019} + i^{2020}.$$

- (A) i (B) 1 (C) 0 (D) -1 (E) $-i$

5. Four distinct straight lines pass through the origin of the coordinate system and intersect the parabola $y = x^2 - 3$ at eight points. What is the product of the x -coordinates of these eight points?

- (A) -81 (B) -9 (C) 0 (D) 9 (E) 81

6. For how many integers n is $|n^2 + 81n - 2020|$ a prime number?

- (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

7. Lockers are numbered with consecutive positive integers beginning with 1, and the digit 2 is used exactly 106 times (including on the last locker). What is the number of the last locker?

- (A) 240 (B) 245 (C) 250 (D) 255 (E) 260

8. * What is the area enclosed by the graph of

$$|3x| + |4y| = 12?$$

- (A) 22 (B) 24 (C) 26 (D) 28 (E) 30

9. Let $s, t,$ and r be the solutions of the equation

$$3x^3 - 4x^2 + 9x - 6 = 0.$$

Find $\frac{1}{s} + \frac{1}{t} + \frac{1}{r}$.

- (A) $-\frac{5}{2}$ (B) $-\frac{3}{2}$ (C) $-\frac{1}{2}$ (D) $\frac{1}{2}$ (E) $\frac{3}{2}$

10. * Let sets $A = \{1, 2, 3, \dots, 99\}$, $B = \{2x | x \in A\}$, and $C = \{x/2 | x \in A\}$. What is the number of elements of $B \cap C$?

- (A) 24 (B) 48 (C) 12 (D) 25 (E) 30

11. Three different numbers are chosen from the set $\{1, 2, 3, \dots, 8\}$. What is the probability that one of them is the average of the other two?

(A) $\frac{2}{7}$ (B) $\frac{1}{8}$ (C) $\frac{9}{56}$ (D) $\frac{5}{28}$ (E) $\frac{3}{14}$

12. * For each positive integer n , the quadratic function $f(x) = ax^2 + bx + c$ satisfies the equation

$$f(1) + f(2) + \dots + f(n) = n^3.$$

What is the value of the product abc ?

(A) -12 (B) -10 (C) -9 (D) -8 (E) -6

13. How many real solutions does the following equation have

$$x^2 + \sqrt{x-1} = 1 + \sqrt{x^2 + x - 2}?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

14. * The quadratic function $f(x) = x^2 - 6x + 2$ intersects the x -axis and the y -axis at three different points. The circle through these three points intersects the graph of f at a fourth point (a, b) . Find the sum $a + b$.

(A) 14 (B) 12 (C) 10 (D) 8 (E) 6

15. Find the number of solutions of the equation $\log(x^2 + 1) = \log_3 x$.

(A) 0 (B) 1 (C) 2
(D) 3 (E) infinitely many

16. The real numbers x, y , and z satisfy the equation

$$\frac{2019}{x+1} + \frac{2019}{y+1} + \frac{2019}{z+1} = 2020.$$

Find the value of

$$\frac{x-1}{x+1} + \frac{y-1}{y+1} + \frac{z-1}{z+1}.$$

- (A) $\frac{2014}{2019}$ (B) $\frac{2017}{2019}$ (C) $\frac{2012}{2019}$ (D) $\frac{2018}{2019}$ (E) $\frac{2015}{2019}$

17. * Find the number of pairs of positive integers (a, b) such that the sum of a and b is a multiple of 4 and $a^2 - b^2 = 2020$.

- (A) 4 (B) 3 (C) 2 (D) 1 (E) 0

18. The real number x satisfies the equation

$$\sin x - \cos x = \frac{1}{3}.$$

Find the value of $\sin(2x)$.

- (A) $\frac{2}{9}$ (B) $\frac{4}{9}$ (C) $\frac{5}{9}$ (D) $\frac{7}{9}$ (E) $\frac{8}{9}$

19. On January 1st, 2020 the Golden Goose laid one golden egg. Every day after January 1st, it laid two more eggs than it did on the day before. How many golden eggs will have the Golden Goose lay this year by the end of the day on March 7, 2020?

- (A) 4096 (B) 4225 (C) 4356 (D) 4489 (E) 4624

20. Find the smallest value of the integer n such that a convex polygon with n sides has at least 2020 diagonals.

- (A) 64 (B) 65 (C) 66 (D) 67 (E) 68

21. * The number

$$\frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \dots}}}}$$

can be written in the form $\frac{A + \sqrt{B}}{2}$, where A and B are integers. Find the value of $A + B$

- (A) 10 (B) 11 (C) 12 (D) 13 (E) 14

22. Let f be a function such that $f(x+y) = f(x)f(y)$, for all integers x and y . If $f(1) = 3$, find the value of

$$f(0) + f(1) + \dots + f(9).$$

- (A) 29524 (B) 29424 (C) 29324 (D) 29224 (E) 29124

23. How many real solutions does the equation

$$||5^x - 4| - 3| = 1$$

have? (Here, $|a|$ means the absolute value of a .)

- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

24. If $f(x) = \frac{ax + b}{cx + d}$, $abcd \neq 0$ and $f(f(x)) = x$, for all x in the domain of f , what is the value of $a + d$?

- (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

25. * Let $a \circ b = \frac{a + b}{ab + 4}$. What is the value of $((\dots((2020 \circ 2019) \circ 2018) \circ \dots \circ 1) \circ 0)$?

- (A) $\frac{1}{20}$ (B) $\frac{1}{505}$ (C) $\frac{1}{1010}$ (D) $\frac{1}{12}$ (E) $\frac{1}{2020}$

26. When expanded as a decimal, the fraction $\frac{1}{19}$ has a repetend (the repeating part of the decimal) that begins right after the decimal point and is 18 digits long. If the last three digits of the repetend are $x21$, what is the digit x ?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

27. * Find the sum of the number $N = 12345$ and all other five digit numbers obtained by permuting the digits of N in all possible ways.

- (A) 4999960 (B) 399960 (C) 999900
(D) 3999960 (E) 9999600

28. Let $f_1(x) = \sqrt{1-x}$, and for integers $n \geq 2$, we define

$$f_n(x) = f_{n-1}(\sqrt{n^2-x}).$$

What is the largest value of n for which the domain of f_n is nonempty?

- (A) 6 (B) 5 (C) 4 (D) 3 (E) 2

29. * Jane has an infinite number of balls and empty boxes available to her. The empty boxes, each capable of holding five balls, are arranged in a row from left to right. At the first step, she places a ball in the first box of the row. At each subsequent step, she places a ball in the first box of the row that still has room for a ball and empties any previous boxes. How many balls in total are in the boxes as a result of Jane's 2020th step?

- (A) 1 (B) 4 (C) 5 (D) 10 (E) 20

30. A right triangle $\triangle ABC$ has integer side lengths. If \overline{BC} is its hypotenuse and $2AC = AB + BC$, which of the following is a possible length for one of its legs?

- (A) 31 (B) 41 (C) 51 (D) 61 (E) 71

31. Suppose $a > 0$, $b > 0$, $c > 0$ and $abc = 1$. What is the minimum value of

$$(a + b)^3 + (b + c)^3 + (c + a)^3?$$

- (A) 8 (B) 16 (C) 24 (D) 32 (E) 40

32. Consider the function

$$f(x) = \sin\left(2x + \frac{3\pi}{2}\right) - 3\cos x.$$

If x varies in the set of real numbers, what is the minimum value of $f(x)$?

- (A) 0 (B) -1 (C) -2 (D) -3 (E) -4

33. * For each positive integer n , we define

$$a_n = \sin(\pi\sqrt{n^2 + 5}).$$

Find $\lim_{n \rightarrow \infty} a_n$

- (A) ∞ (B) 5 (C) 0
(D) 1 (E) does not exist

34. If $a < b < c$ are consecutive prime numbers and the equation $ax^2 + cx + b = 0$ has an integer root, how many values are possible for b ?

- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

35. * Consider the function f defined by $f(z) = (4+i)z^2 + \alpha z + \beta$, for all complex numbers z , where α and β are complex numbers and $i^2 = -1$. Suppose that $f(1)$ and $f(i)$ are both real. What is the smallest possible value of $|\alpha| + |\beta|$?

- (A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2 (D) $\sqrt{5}$ (E) $\sqrt{6}$

36. If a , b , and c are real numbers such that

$$\lim_{x \rightarrow \infty} x(\sqrt{ax^2 + bx + c} - 2x - 1) = 10,$$

find the value of $a + b + c$.

- (A) 41 (B) 45 (C) 49 (D) 53 (E) 57

37. * Let $(a_n)_{n \geq 0}$ be the sequence defined by the formula

$$a_n = 1 + \frac{1}{2^{2^n}},$$

for any nonnegative integer n . Which of the following is equal to the value of the product $a_0 a_1 a_2 \cdots a_{10}$?

- (A) $2 + \frac{1}{2^{2^{10}+1}}$ (B) $2 + \frac{1}{2^{2^9+1}}$ (C) $2 + \frac{1}{2^{2^9-1}}$
(D) $2 - \frac{1}{2^{2^{10}+1}}$ (E) $2 - \frac{1}{2^{2^{11}-1}}$

38. * If $x_1, x_2, \dots, x_{2020}$ are equal to either -2 or 3 , which of the following integers M can be written as

$$M = x_1 + 2x_2 + 3x_3 + \cdots + 2020x_{2020}?$$

- (A) 2016 (B) 2017 (C) 2018 (D) 2019 (E) 2020

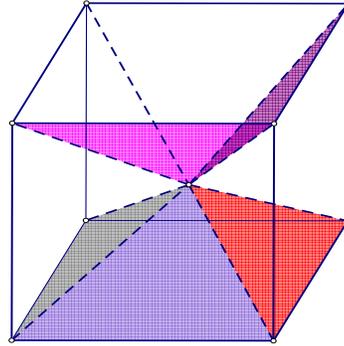
39. * Find the number of solutions of the equation

$$\tan\left(\frac{x}{2}\right) = \cot(3x),$$

in the interval $[0, 2\pi]$

- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

40. A cube is dissected into six pyramids by connecting a given point in the interior of the cube with each vertex of the cube. The volumes of five of these pyramids are 1, 4, 9, 10 and 13. What is the volume of the sixth pyramid?



- (A) 4 (B) 5 (C) 6
(D) 7 (E) 8

41. * Find the number of pairs (x, y) of positive integers such that

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{2020}.$$

- (A) 20 (B) 22 (C) 24 (D) 26 (E) 28

42. * For a set of n prime numbers, if the sum of any three different primes in the set is also prime, what is the largest possible value of n ?

- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

43. Let $a, b, c, x, y,$ and z be nonzero complex numbers such that

$$a = \frac{b+c}{x+1}, \quad b = \frac{c+a}{y+1}, \quad c = \frac{a+b}{z+1}.$$

If $xy + yz + zx = 200$ and $x + y + z = 100$, find the value of xyz .

- (A) -196 (B) -192 (C) -188 (D) -184 (E) -180

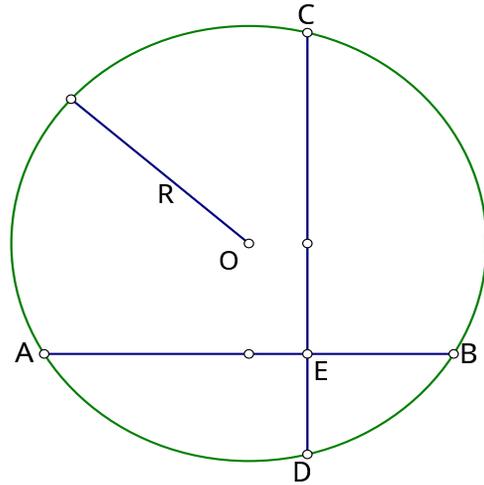
44. * How many ordered pairs of positive integers (x, y) , with $x < y$, satisfy the equation

$$x^2 + y^2 = 2020?$$

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

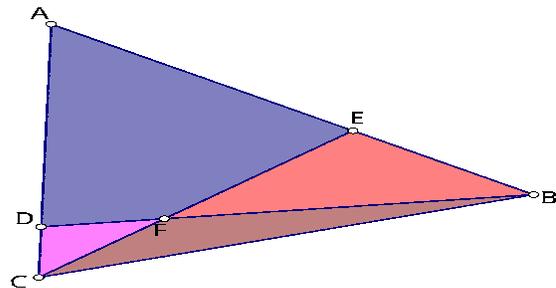
45. * Two chords \overline{AB} and \overline{CD} , in a circle of radius R , are perpendicular at E (see the Figure). Knowing that $AE = 9$, $EB = 5$ and $EC = 12$, find R .

- (A) $\frac{65}{8}$ (B) $\frac{63}{8}$ (C) $\frac{61}{8}$
 (D) $\frac{59}{8}$ (E) $\frac{57}{8}$



46. * In the adjacent Figure, we have two Cevians in the triangle ABC , \overline{CE} and \overline{BD} intersecting at F . Knowing that the areas of the triangles DFC , CFB and FBE are equal to 2, 6 and 9 respectively, find the area of the quadrilateral $ADFE$.

- (A) 20 (B) 21 (C) 22
 (D) 23 (E) 24



47. For which value(s) of the real number a does the function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

admit an antiderivative? That is, for which value(s) of a there exist a differentiable function F , such that $F'(x) = f(x)$, for all real numbers x ?

- (A) $(-\infty, \infty)$ (B) $[-1, 1]$ (C) 0
 (D) -1 and 1 (E) none

48. Let a, b, c be the real roots of $x^3 - 4x^2 - 32x + 17 = 0$. The solution of the equation

$$\sqrt[3]{x-a} + \sqrt[3]{x-b} + \sqrt[3]{x-c} = 0$$

can be written as $\frac{M}{N}$, with the fraction in lowest terms. What is the value of $M + N$?

- (A) 1531 (B) 1532 (C) 1533 (D) 1534 (E) 1535

49. What is the value of the limit

$$\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right]?$$

- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$ (E) $\frac{\pi}{8}$

50. Two people agree to meet at a coffee shop. They each independently pick a random moment in time between 9 a.m. and 10 a.m. and show up exactly at their selected time. But they are very impatient, and only stay for 5 minutes after when they arrive. What is the probability that they meet?

- (A) $\frac{19}{144}$ (B) $\frac{23}{144}$ (C) $\frac{29}{144}$ (D) $\frac{31}{144}$ (E) $\frac{37}{144}$