

**Solutions of the Forty-Sixth Annual Columbus State Invitational Mathematics
Tournament**

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1. How many positive divisors does 2020 have?

- (A) 9 (B) 10 (C) 11 (D) 12 (E) 13

Answer: D

We have that $2020 = 2^2 \cdot 5 \cdot 101$, so the number of positive divisors is 12.

2. For what values of the real number a does the equation

$$x^2 - ax - a = 0$$

have two different complex zeros, which are not real numbers?

- (A) $a > 0$ (B) $a < 0$ (C) $0 < a < 4$
(D) $-4 < a < 0$ (E) $a < 4$

Answer: D

In order for the equation to have complex zeros, which are not real, we must have $a^2 + 4a < 0$. This implies that $-4 < a < 0$.

3. Let x and y be positive integers such that

$$\frac{2020!}{5^x \cdot 11^y}$$

is an integer. What is the largest possible value of $x + y$?

- (A) 695 (B) 697 (C) 699 (D) 701 (E) 703

Answer: E

The exponent of 5 in $2020!$ is $404+80+16+3=503$ and the exponent of 11 is $183+16+1=200$, so $x + y = 703$. Here, 183 represents the number of multiples of 5 smaller or equal to 2020, 80 represents the number multiples of 25, 16 represents all multiples of 125, and 3 is the number of multiples of 625. Similarly, by using the powers of 11, we get the exponent of 11 in $2020!$

4. * If $i = \sqrt{-1}$, find the sum

$$1 + i + i^2 + i^3 + \dots + i^{2019} + i^{2020}.$$

- (A) i (B) 1 (C) 0 (D) -1 (E) $-i$

Answer: B

For each nonnegative integer k , we have that $i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1$, and $i^{4k+3} = -i$. This implies that $i^{4k} + i^{4k+1} + i^{4k+2} + i^{4k+3} = 0$, so the sum is equal to $i^{2020} = 1$.

5. Four distinct straight lines pass through the origin of the coordinate system and intersect the parabola $y = x^2 - 3$ at eight points. What is the product of the x -coordinates of these eight points?

- (A) -81 (B) -9 (C) 0 (D) 9 (E) 81

Answer: E

Any line as in the problem has equation of the form $y = mx$. The x -coordinates of the points of intersection with the parabola satisfy the equation $x^2 - 3 = mx$, which is equivalent to $x^2 - mx - 3 = 0$. By Vieta's formulas, the product of the solutions of this equation is -3 . For the four lines, we get a product of $(-3)^4 = 81$.

6. For how many integers n is $|n^2 + 81n - 2020|$ a prime number?

- (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Answer: A

We have $|n^2 + 81n - 2020| = |(n + 101)(n - 20)|$. This implies that $n + 101 = \pm 1$ or $n - 20 = \pm 1$. If $n + 101 = 1$, we have $|(n + 101)(n - 20)| = 120$. If $n + 101 = -1$, then $|(n + 101)(n - 20)| = 122$. If $n - 20 = 1$, then $|(n + 101)(n - 20)| = 122$. Finally, if $n - 20 = -1$, then $|(n + 101)(n - 20)| = 120$. In conclusion, there are no integers n such that $|n^2 + 81n - 2020|$ is a prime number.

7. Lockers are numbered with consecutive positive integers beginning with 1, and the digit 2 is used exactly 106 times (including on the last locker). What is the number of the last locker?

(A) 240 (B) 245 (C) 250 (D) 255 (E) 260

Answer: C

The digit 2 is used 20 times for the lockers 1-99 and 100-199. The digit 2 is used 11 times for each of the lockers 200-209; 210-219; 230-239; 240-249. It is also used 21 times for lockers 220-229. So the digit 2 is used 105 times for the lockers 1-249, which implies that the last locker is 250.

8. * What is the area enclosed by the graph of

$$|3x| + |4y| = 12?$$

(A) 22 (B) 24 (C) 26 (D) 28 (E) 30

Answer: B

We have to compute the area of the quadrilateral with vertices $(4, 0)$, $(0, 3)$, $(-4, 0)$, and $(0, -3)$. The diagonals of this quadrilateral are perpendicular, so its area is half of their product. This implies that the area is 24.

9. Let s, t , and r be the solutions of the equation

$$3x^3 - 4x^2 + 9x - 6 = 0.$$

Find $\frac{1}{s} + \frac{1}{t} + \frac{1}{r}$.

(A) $-\frac{5}{2}$ (B) $-\frac{3}{2}$ (C) $-\frac{1}{2}$ (D) $\frac{1}{2}$ (E) $\frac{3}{2}$

Answer: E

By Vieta's formulas, we have $tr + rs + st = 3$ and $str = 2$. This implies that

$$\frac{1}{s} + \frac{1}{t} + \frac{1}{r} = \frac{tr + rs + st}{str} = \frac{3}{2}.$$

10. * Let sets $A = \{1, 2, 3, \dots, 99\}$, $B = \{2x | x \in A\}$, and $C = \{x/2 | x \in A\}$. What is the number of elements of $B \cap C$?
- (A) 24 (B) 48 (C) 12 (D) 25 (E) 30

Answer: A

We have that $B = \{2, 4, 6, \dots, 198\}$ and $C = \{1/2, 1, 3/2, 2, \dots, 49, 99/2\}$. This implies that $B \cap C = \{2, 4, 6, \dots, 48\}$, so the number of elements is 24.

11. Three different numbers are chosen from the set $\{1, 2, 3, \dots, 8\}$. What is the probability that one of them is the average of the other two?
- (A) $\frac{2}{7}$ (B) $\frac{1}{8}$ (C) $\frac{9}{56}$ (D) $\frac{5}{28}$ (E) $\frac{3}{14}$

Answer: E

There are $\binom{8}{3} = 56$ ways to choose the three numbers, $a > b > c$. We must have $a + c = 2b$, which implies that $a + c$ is even. We get the following possibilities $(8, 7, 6)$, $(8, 6, 4)$, $(8, 5, 2)$, $(7, 6, 5)$, $(7, 5, 3)$, $(7, 4, 1)$, $(6, 5, 4)$, $(6, 4, 2)$, $(5, 4, 3)$, $(5, 3, 1)$, $(4, 3, 2)$, $(3, 2, 1)$. Therefore, the probability is $12/56 = 3/14$.

12. * For each positive integer n , the quadratic function $f(x) = ax^2 + bx + c$ satisfies the equation

$$f(1) + f(2) + \dots + f(n) = n^3.$$

What is the value of the product abc ?

- (A) -12 (B) -10 (C) -9 (D) -8 (E) -6

Answer: C

We have that $f(1) = a + b + c = 1$, $f(2) = (f(2) + f(1)) - f(1) = 8 - 1 = 7 = 4a + 2b + c$, and $f(3) = (f(3) + f(2) + f(1)) - (f(2) + f(1)) = 27 - 8 = 19 = 9a + 3b + c$. Subtracting the first equation from the second and also the second from the third, we can eliminate c . We get $3a + b = 6$ and $5a + b = 12$, which imply $a = 3$, $b = -3$, and $c = 1$, so $abc = -9$.

13. How many real solutions does the following equation have

$$x^2 + \sqrt{x-1} = 1 + \sqrt{x^2 + x - 2}?$$

- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: C

The conditions for the existence of the radicals imply that $x \geq 1$. The equation can be rewritten as

$$(x-1)(x+1) = \sqrt{(x-1)(x+2)} - \sqrt{x-1}.$$

Clearly $x = 1$ is a solution. Assuming that $x > 1$, we can divide by $\sqrt{x-1}$ and we get the equation

$$(x+1)\sqrt{x-1} = \sqrt{x+2} - 1.$$

This implies that

$$(x+1)\sqrt{x-1} = \frac{x+1}{\sqrt{x+2}+1}.$$

Since $x > 1$, we can simplify this to

$$\sqrt{x-1} = \frac{1}{\sqrt{x+2}+1}.$$

For $x > 1$, the left side of the equation is an increasing function with range $(0, \infty)$ and the right side is a decreasing function with range $(0, 1/(\sqrt{3}+1))$, so the equation has an unique solution for $x > 1$. This means that the original equation has two real solutions.

14. * The quadratic function $f(x) = x^2 - 6x + 2$ intersects the x -axis and the y -axis at three different points. The circle through these three points intersects the graph of f at a fourth point (a, b) . Find the sum $a + b$.
- (A) 14 (B) 12 (C) 10 (D) 8 (E) 6

Answer: D

Since the circle passes through the x -intercepts, its center is on the vertical line $x = 3$, which passes through the vertex of the parabola. This implies that the 4th point of intersection is the symmetric of $(0, 2)$ with respect to the line $x = 3$. So the point is of the form $(a, 2)$. Because the point is on the parabola, we have $a^2 - 6a + 2 = 2$, so $a = 0$ or $a = 6$. Since $(0, 2)$ is the y -intercept, the 4th point is $(6, 2)$ so $a + b = 8$.

15. Find the number of solutions of the equation $\log(x^2 + 1) = \log_3 x$.

- (A) 0 (B) 1 (C) 2
(D) 3 (E) infinitely many

Answer: B

Consider the function, given by the above equation, $f(x) = \frac{\ln(x^2 + 1)}{\ln 10} - \frac{\ln x}{\ln 3}$, where $x > 0$. The function is differentiable and its derivative is

$$f'(x) = \frac{2x}{(x^2 + 1) \ln 10} - \frac{1}{x \ln 3} = \frac{x^2(\ln 9 - \ln 10) - \ln 10}{(\ln 3)(\ln 10)x(x^2 + 1)} < 0, \text{ for all } x > 0.$$

This implies that f is decreasing, so the equation $f(x) = 0$ has at most one solution. On the other hand, it is easy to see that $x = 3$ is a solution. Therefore the given equation has only one solution.

16. The real numbers x, y , and z satisfy the equation

$$\frac{2019}{x+1} + \frac{2019}{y+1} + \frac{2019}{z+1} = 2020.$$

Find the value of

$$\frac{x-1}{x+1} + \frac{y-1}{y+1} + \frac{z-1}{z+1}.$$

- (A) $\frac{2014}{2019}$ (B) $\frac{2017}{2019}$ (C) $\frac{2012}{2019}$ (D) $\frac{2018}{2019}$ (E) $\frac{2015}{2019}$

Answer: B

We have that

$$\frac{x-1}{x+1} + \frac{y-1}{y+1} + \frac{z-1}{z+1} = 1 - \frac{2}{x+1} + 1 - \frac{2}{y+1} + 1 - \frac{2}{z+1},$$

which simplifies to

$$3 - 2 \cdot \frac{2020}{2019} = \frac{2017}{2019}.$$

17. * Find the number of pairs of positive integers (a, b) such that the sum of a and b is a multiple of 4 and $a^2 - b^2 = 2020$.
- (A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Answer: E

The equation is equivalent to $(a - b)(a + b) = 2^2 \cdot 5 \cdot 101$. Since a and b are supposed to be integers, we must have that $a + b$ and $a - b$ have the same parity. However, if $a + b$ is a multiple of 4, then $a - b$ would have to be a multiple of 2, so $(a - b)(a + b)$ would have to be a multiple of 8. This is impossible since 2020 is not a multiple of 8. In conclusion, the number of solutions is 0.

18. The real number x satisfies the equation

$$\sin x - \cos x = \frac{1}{3}.$$

Find the value of $\sin(2x)$.

- (A) $\frac{2}{9}$ (B) $\frac{4}{9}$ (C) $\frac{5}{9}$ (D) $\frac{7}{9}$ (E) $\frac{8}{9}$

Answer: E

If we square the equation, we get

$$\sin^2 x - 2 \sin x \cos x + \cos^2 x = \frac{1}{9}.$$

Using that $\sin(2x) = 2 \sin x \cos x$ and $\sin^2 x + \cos^2 x = 1$, we get that $\sin(2x) = \frac{8}{9}$.

19. On January 1st, 2020 the Golden Goose laid one golden egg. Every day after January 1st, it laid two more eggs than it did on the day before. How many golden eggs will have the Golden Goose lay this year by the end of the day on March 7, 2020?
- (A) 4096 (B) 4225 (C) 4356 (D) 4489 (E) 4624

Answer: D

There are 31 days in January and 29 in February. This means that March 7, 2020 is the 67th day of the year, so we must compute

$$1 + 3 + 5 + \cdots + 133 = \frac{134 \cdot 67}{2} = 4489.$$

20. Find the smallest value of the integer n such that a convex polygon with n sides has at least 2020 diagonals.

- (A) 64 (B) 65 (C) 66 (D) 67 (E) 68

Answer: C

Each vertex can be connected with $n - 3$ vertices for form a diagonal, thus the number of diagonals of a convex polygon with n sides is $\frac{n(n - 3)}{2}$. We must find the smallest positive integer n such that $\frac{n(n - 3)}{2} \geq 2020$. It is easy to see that $n = 66$.

21. * The number

$$\frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \dots}}}}$$

can be written in the form $\frac{A + \sqrt{B}}{2}$, where A and B are integers. Find the value of $A + B$

- (A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Answer: A

Denote the number by x and note that we have the equation

$$x = \frac{1}{1 + \frac{1}{2+x}}$$

The equation simplifies to $x = \frac{2 + x}{3 + x}$. Thus we have $x^2 + 3x = 2 + x$, which gives us $x^2 + 2x - 2 = 0$. Since x is positive, we get $x = (-2 + \sqrt{12})/2$. This implies that $A + B = 10$.

22. Let f be a function such that $f(x + y) = f(x)f(y)$, for all integers x and y . If $f(1) = 3$, find the value of

$$f(0) + f(1) + \dots + f(9).$$

- (A) 29524 (B) 29424 (C) 29324 (D) 29224 (E) 29124

Answer: A

If $x = 0$ and $y = 1$, we get that $f(1) = f(0)f(1)$, so $f(0) = 1$. For $x = y = 1$, we get $f(2) = f(1)f(1) = 9$ and, in general, $f(n) = 3^n$, for all nonnegative integers n . Thus,

$$f(0) + f(1) + \cdots + f(9) = 1 + 3 + 3^2 + \cdots + 3^9 = \frac{3^{10} - 1}{2} = 29524.$$

23. How many real solutions does the equation

$$||5^x - 4| - 3| = 1$$

have? (Here, $|a|$ means the absolute value of a .)

- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: D

We have $|5^x - 4| = 4$ or $|5^x - 4| = 2$. In the first case, we have $5^x = 0$ or $5^x = 8$, and only the second equation has solutions. In the second case, we have $5^x = 6$ or $5^x = 2$ and both equations have solutions. In conclusion, we have three solutions ($\log_5 8$, $\log_5 6$, and $\log_5 2$).

24. If $f(x) = \frac{ax + b}{cx + d}$, $abcd \neq 0$ and $f(f(x)) = x$, for all x in the domain of f , what is the value of $a + d$?

- (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Answer: A

Note that the function f is its own inverse. The vertical asymptote of f is $x = -d/c$ and the horizontal asymptote is $y = a/c$. The vertical asymptote of the inverse function is $x = a/c$ and the horizontal asymptote is $y = -d/c$. Since $f = f^{-1}$ we have that $-d/c = a/c$, so $a + d = 0$.

25. * Let $a \circ b = \frac{a + b}{ab + 4}$. What is the value of $((\cdots((2020 \circ 2019) \circ 2018) \circ \cdots \circ 1) \circ 0)$?

- (A) $\frac{1}{20}$ (B) $\frac{1}{505}$ (C) $\frac{1}{1010}$ (D) $\frac{1}{12}$ (E) $\frac{1}{2020}$

Answer: D

Note that $a \circ 2 = \frac{a + 2}{2a + 4} = \frac{1}{2}$, so the entire expression is equal to $\left(\frac{1}{2} \circ 1\right) \circ 0 = \frac{1}{3} \circ 0 = \frac{1}{12}$.

26. When expanded as a decimal, the fraction $\frac{1}{19}$ has a repetend (the repeating part of the decimal) that begins right after the decimal point and is 18 digits long. If the last three digits of the repetend are $x21$, what is the digit x ?
- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: D

We have the equation

$$\frac{1}{19} = \frac{\dots x21}{10^{18} - 1},$$

so $19 \cdot (\dots x21) = 10^{18} - 1$. Since the last three digits of $19 \cdot (\dots x21)$ must be nines, we get that $x = 4$.

27. * Find the sum of the number $N = 12345$ and all other five digit numbers obtained by permuting the digits of N in all possible ways.
- (A) 4999960 (B) 399960 (C) 999900
- (D) 3999960 (E) 9999600

Answer: D

There are 120 numbers that we need to add. Consider $N = 12345$ and its circular permutations: 23451, 34512, 45123, and 51234. The sum of these numbers is

$$(1 + 2 + 3 + 4 + 5)(10^4 + 10^3 + 10^2 + 10^1 + 1) = 15 \cdot \frac{10^5 - 1}{9}.$$

There are 24 families of circular permutations that we can form and each produces the same sum. Therefore, the sum of all these numbers is $24 \cdot 15 \cdot \frac{10^5 - 1}{9} = 3999960$.

28. Let $f_1(x) = \sqrt{1 - x}$, and for integers $n \geq 2$, we define

$$f_n(x) = f_{n-1}(\sqrt{n^2 - x}).$$

What is the largest value of n for which the domain of f_n is nonempty?

- (A) 6 (B) 5 (C) 4 (D) 3 (E) 2

Answer: B

The function

$$f_2(x) = f_1(\sqrt{4-x}) = \sqrt{1-\sqrt{4-x}}$$

has domain $[3, 4]$. We also have $f_3(x) = f_2(\sqrt{9-x})$. In order for f_3 to be defined we must have $3 \leq \sqrt{9-x} \leq 4$, so the domain of f_3 is $[-7, 0]$. Similarly, $f_4(x) = f_3(\sqrt{16-x})$ and in order to be defined, we must have $x \leq 16$, and $\sqrt{16-x}$ must lie in the interval $[-7, 0]$. But $\sqrt{16-x}$ is always nonnegative, so we must have $\sqrt{16-x} = 0$, or $x = 16$. Thus, the domain of f_4 consists of a single point $\{16\}$. However, 4 is not the largest value of n . We see, then, that $f_5(x) = f_4(\sqrt{25-x})$ is defined if and only if $\sqrt{25-x} = 16$, or $x = 25 - 16^2 = -231$. Therefore, the domain of f_5 is $\{-231\}$. The domain of $f_6(x)$ is empty, because $\sqrt{36-x}$ can never equal a negative number like -231 . Thus, the largest value of n is 5.

29. * Jane has an infinite number of balls and empty boxes available to her. The empty boxes, each capable of holding five balls, are arranged in a row from left to right. At the first step, she places a ball in the first box of the row. At each subsequent step, she places a ball in the first box of the row that still has room for a ball and empties any previous boxes. How many balls in total are in the boxes as a result of Jane's 2020th step?

- (A) 1 (B) 4 (C) 5 (D) 10 (E) 20

Answer: D

The boxes resemble the set of positive integers in base 6. In particular, the first box corresponds to the units digit, the second corresponds to the sixes digit, and so forth. An empty box corresponds to the digit 0 and a box with k balls, $1 \leq k \leq 5$, corresponds to the digit k . It follows that the number of filled boxes on the 2020th step is just the sum of the digits in the base 6 expression of 2020. We have $2020 = 1 \cdot 6^4 + 3 \cdot 6^3 + 2 \cdot 6^2 + 4$, so the number of balls is 10.

30. A right triangle $\triangle ABC$ has integer side lengths. If \overline{BC} is its hypotenuse and $2AC = AB + BC$, which of the following is a possible length for one of its legs?

- (A) 31 (B) 41 (C) 51 (D) 61 (E) 71

Answer: C

With the standard notations, we have the equations $a^2 = b^2 + c^2$ and $2b = c + a$. Using that $a = 2b - c$, we get that $3b^2 - 4bc = 0$, so $3b = 4c$. Since b and c are integers, we have that b is a multiple of 4 and c is a multiple of 3. This implies that $c = 51$.

31. Suppose $a > 0$, $b > 0$, $c > 0$ and $abc = 1$. What is the minimum value of

$$(a+b)^3 + (b+c)^3 + (c+a)^3?$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \sin(\pi\sqrt{n^2+5}) &= \lim_{n \rightarrow \infty} (-1)^n \sin(\pi\sqrt{n^2+5} - n\pi) = \\ &= \lim_{n \rightarrow \infty} (-1)^n \sin\left[\frac{5\pi}{\sqrt{n^2+5}+n}\right] = 0.\end{aligned}$$

34. If $a < b < c$ are consecutive prime numbers and the equation $ax^2 + cx + b = 0$ has an integer root, how many values are possible for b ?
- (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: B

If $a > 2$, then a, b , and c are all odd. In this case, the equation can't have an integer solution because $ax^2 + cx + b$ is odd for all values of the integer x . This implies that $a = 2$ and, since $a < b < c$ are consecutive prime numbers, we get that $b = 3$ and $c = 5$. The equation $2x^2 + 5x + 3 = 0$ has the integer solution $x = -1$. Thus, there is only one possible value of b with the required property.

35. * Consider the function f defined by $f(z) = (4+i)z^2 + \alpha z + \beta$, for all complex numbers z , where α and β are complex numbers and $i^2 = -1$. Suppose that $f(1)$ and $f(i)$ are both real. What is the smallest possible value of $|\alpha| + |\beta|$?
- (A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2 (D) $\sqrt{5}$ (E) $\sqrt{6}$

Answer: A

Let $\alpha = a + bi$ and $\beta = c + di$, where a, b, c , and d are real numbers. Then

$$\begin{aligned}f(1) &= (4+i) + \alpha + \beta = (a+c+4) + (b+d+1)i, \\ f(i) &= (4+i)(-1) + \alpha i + \beta = (-b+c-4) + (a+d-1)i.\end{aligned}$$

Since $f(1)$ and $f(i)$ are both real, $b+d+1 = 0$ and $a+d-1 = 0$, so $a = -d+1$ and $b = -d-1$. Then

$$\begin{aligned}|\alpha| + |\beta| &= \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \\ &= \sqrt{(-d+1)^2 + (-d-1)^2} + \sqrt{c^2 + d^2} \\ &= \sqrt{2d^2 + 2} + \sqrt{c^2 + d^2} \\ &\geq \sqrt{2}.\end{aligned}$$

Equality occurs when $a = 1$, $b = -1$, $c = 0$, and $d = 0$. Therefore, the minimum value is $\sqrt{2}$.

36. If a, b , and c are real numbers such that

$$\lim_{x \rightarrow \infty} x(\sqrt{ax^2 + bx + c} - 2x - 1) = 10,$$

find the value of $a + b + c$.

- (A) 41 (B) 45 (C) 49 (D) 53 (E) 57

Answer: C

We have that

$$\lim_{x \rightarrow \infty} x(\sqrt{ax^2 + bx + c} - 2x - 1) = \lim_{x \rightarrow \infty} \frac{x(ax^2 + bx + c - 4x^2 - 4x - 1)}{\sqrt{ax^2 + bx + c} + 2x + 1} = 10.$$

Since the dominant term on the denominator is x , in order for the limit to be finite, we need to have $a = 4$ and $b = 4$, so that the cubic and quadratic terms of the numerator cancel. Thus we get the equation

$$\lim_{x \rightarrow \infty} \frac{x(c - 1)}{x \left(\sqrt{4 + \frac{4}{x} + \frac{c}{x^2}} + 2 + \frac{1}{x} \right)} = \lim_{x \rightarrow \infty} \frac{c - 1}{\sqrt{4 + \frac{4}{x} + \frac{c}{x^2}} + 2 + \frac{1}{x}} = \frac{c - 1}{4} = 10.$$

This implies that $c = 41$, so $a + b + c = 49$.

37. * Let $(a_n)_{n \geq 0}$ be the sequence defined by the formula

$$a_n = 1 + \frac{1}{2^{2^n}},$$

for any nonnegative integer n . Which of the following is equal to the value of the product $a_0 a_1 a_2 \cdots a_{10}$?

- (A) $2 + \frac{1}{2^{2^{10}+1}}$ (B) $2 + \frac{1}{2^{2^9+1}}$ (C) $2 + \frac{1}{2^{2^9-1}}$
(D) $2 - \frac{1}{2^{2^{10}+1}}$ (E) $2 - \frac{1}{2^{2^{11}-1}}$

Answer: E

Note that $\left(1 - \frac{1}{2^{2^n}}\right) \left(1 + \frac{1}{2^{2^n}}\right) = 1 - \frac{1}{2^{2^{n+1}}}$. We have to simplify

$$a_0 a_1 a_2 \cdots a_{10} = \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{2^2}\right) \left(1 + \frac{1}{2^4}\right) \cdots \left(1 + \frac{1}{2^{2^{10}}}\right).$$

If we multiply by $1/2$, we get

$$\left(1 - \frac{1}{2}\right) a_0 a_1 a_2 \cdots a_{10} = \left(1 - \frac{1}{2}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{2^2}\right) \left(1 + \frac{1}{2^4}\right) \cdots \left(1 + \frac{1}{2^{2^{10}}}\right)$$

$$\begin{aligned}
&= \left(1 - \frac{1}{2^2}\right) \left(1 + \frac{1}{2^2}\right) \left(1 + \frac{1}{2^4}\right) \cdots \left(1 + \frac{1}{2^{2^{10}}}\right) \\
&= \left(1 - \frac{1}{2^4}\right) \left(1 + \frac{1}{2^4}\right) \left(1 + \frac{1}{2^8}\right) \cdots \left(1 + \frac{1}{2^{2^{10}}}\right) = \cdots = 1 - \frac{1}{2^{2^{11}}}.
\end{aligned}$$

This implies that $a_0 a_1 a_2 \cdots a_{10} = 2 \left(1 - \frac{1}{2^{2^{11}}}\right) = 2 - \frac{1}{2^{2^{11}-1}}$.

38. * If $x_1, x_2, \dots, x_{2020}$ are equal to either -2 or 3 , which of the following integers M can be written as

$$M = x_1 + 2x_2 + 3x_3 + \cdots + 2020x_{2020}?$$

- (A) 2016 (B) 2017 (C) 2018 (D) 2019 (E) 2020

Answer: E

The smallest number which can be written in this form is obtained when $x_1 = x_2 = \cdots = x_{2020} = -2$ and is equal to $-2(1 + 2 + 3 + \cdots + 2020) = -2020 \cdot 2021 = -4,082,420$. Note that this number is a multiple of 5. The largest value of M is $3(1 + 2 + 3 + \cdots + 2020) = 6,123,630$. Changing any number of x_i 's from a value of -2 to a 3 results in increasing the value of M by a multiple of 5. It is easy to see that each multiple of 5 between the minimum and the maximum values of M can be obtained this way, so the answer is 2020.

39. * Find the number of solutions of the equation

$$\tan\left(\frac{x}{2}\right) = \cot(3x),$$

in the interval $[0, 2\pi]$

- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

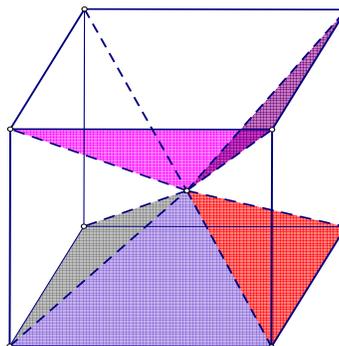
Answer: E

The conditions for the existence of the expressions are $\cos\left(\frac{x}{2}\right) \neq 0$ and $\sin(3x) \neq 0$. Taking these into account, the equation is equivalent with

$$\sin\left(\frac{x}{2}\right) \sin(3x) = \cos\left(\frac{x}{2}\right) \cos(3x)$$

This implies that $\cos\left(3x + \frac{x}{2}\right) = 0$, so $7x/2$ is an odd multiple of $\pi/2$. We get that $x = \pi/7, 3\pi/7, 5\pi/7, \pi, 9\pi/7, 11\pi/7$, and $13\pi/7$. Checking these values against the conditions we see that they all work, so there are 7 solutions.

40. A cube is dissected into six pyramids by connecting a given point in the interior of the cube with each vertex of the cube. The volumes of five of these pyramids are 1, 4, 9, 10 and 13. What is the volume of the sixth pyramid?



- (A) 4 (B) 5 (C) 6
 (D) 7 (E) 8

Answer: B

The sum of the volumes of opposite pyramids is the same since it is $1/3$ of the volume of the cube. This means that the sum of opposite pyramids is 14, and the volume of the 6th pyramid is 5.

41. * Find the number of pairs (x, y) of positive integers such that

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{2020}.$$

- (A) 20 (B) 22 (C) 24 (D) 26 (E) 28

Answer: B

The equation can be written as

$$xy + 2020x - 2020y = 0,$$

which implies that

$$(x - 2020)(y + 2020) = -2020^2.$$

Since y is positive, we have $y + 2020 > 0$, so $x - 2020$ must be a negative factor of $2020^2 = 2^4 \cdot 5^2 \cdot 101^2$. Since x is positive, the only possibilities are $x - 2020 \in \{-1, -2, -4, -8, -16, -5, -10, -20, -40, -80, -25, -50, -100, -200, -400, -101, -202, -404, -808, -1616, -505, -1010\}$. There are 22 solutions.

42. * For a set of n prime numbers, if the sum of any three different primes in the set is also prime, what is the largest possible value of n ?
- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Answer: C

If 2 is one of the primes in the set, then the set can't have more than two primes. Indeed, if it has three or more primes, then 2 plus the sum of any other two primes would be a multiple of 2 and not prime.

If 3 is one of the primes in the set, then the set can't have more than three primes. If the set has four or more primes, then we can't have three primes of the form $3k + 1$, for $k \in \mathbb{Z}$ because if we add these primes we will get a multiple of 3. We also can't have three primes of the form $3k + 2$, for the same reason. We also can't have primes of the form $3k + 1$ and $3k + 2$ because these added together with 3 would be again a multiple of 3. Since any prime greater than 3 is of the form $3k + 1$ or $3k + 2$, we get that the maximum number of elements, in case that 3 is in the set, is three.

If the smallest prime in the set is greater than 3, then the maximum number of primes is four. Indeed, if the set had five or more primes, then at least three of them would be of the form $3k + 1$ or $3k + 2$. By adding them, we would get a number which is not prime. A set with four primes is $\{7, 13, 17, 23\}$.

43. Let $a, b, c, x, y,$ and z be nonzero complex numbers such that

$$a = \frac{b+c}{x+1}, \quad b = \frac{c+a}{y+1}, \quad c = \frac{a+b}{z+1}.$$

If $xy + yz + zx = 200$ and $x + y + z = 100$, find the value of xyz .

- (A) -196 (B) -192 (C) -188 (D) -184 (E) -180

Answer: A

We have that $\frac{b+c}{a} = x+1$. By adding 1 to both sides, we get $\frac{a+b+c}{a} = x+2$, so $\frac{a}{a+b+c} = \frac{1}{x+2}$.

Similarly, we get that $\frac{b}{a+b+c} = \frac{1}{y+2}$ and $\frac{c}{a+b+c} = \frac{1}{z+2}$. This implies that we have the equation

$$\frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} = 1.$$

Using a common denominator, the equation is equivalent to

$$\frac{xy + yz + xz + 4(x + y + z) + 12}{xyz + 2(xy + yz + zx) + 4(x + y + x) + 8} = 1,$$

which implies that $\frac{612}{xyz + 808} = 1$, so $xyz = -196$.

44. * How many ordered pairs of positive integers (x, y) , with $x < y$, satisfy the equation

$$x^2 + y^2 = 2020?$$

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: B

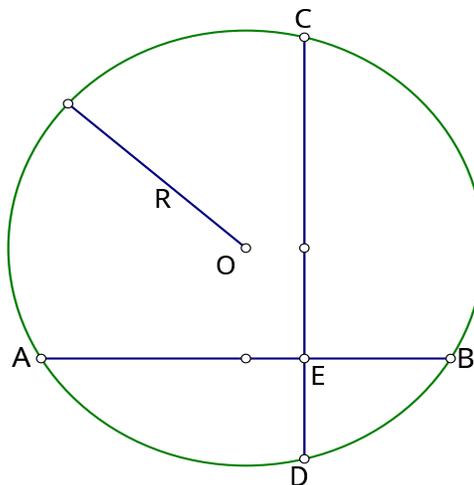
Since $2020 = 2^2 \cdot 5 \cdot 101$, we have that

$$(x + iy)(x - iy) = (1 + i)^2(1 - i)^2(1 + 2i)(1 - 2i)(1 + 10i)(1 - 10i).$$

Using the unique factorization in the ring of Gaussian integers, we get that $x + iy = A \cdot B \cdot C$, where $A = 1 \pm i$, $B = 1 \pm 2i$, and $C = 1 \pm 10i$, for any choice of the plus/minus signs. This implies that we have $x + iy = \pm 24 \pm 38i$ or $\pm 16 \pm 42i$. The solutions which satisfy the requirement $0 < x < y$ are $(24, 38)$ and $(16, 42)$, thus there are two solutions.

45. * Two chords \overline{AB} and \overline{CD} , in a circle of radius R , are perpendicular at E (see the Figure). Knowing that $AE = 9$, $EB = 5$ and $EC = 12$, find R .

- (A) $\frac{65}{8}$ (B) $\frac{63}{8}$ (C) $\frac{61}{8}$
 (D) $\frac{59}{8}$ (E) $\frac{57}{8}$



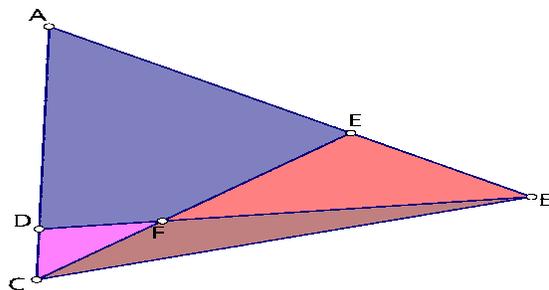
Answer: A

Using the Pythagorean Theorem in the right triangles $\triangle EAC$ and $\triangle EBC$, we get that $AC = 15$ and $BC = 13$. Now, we express the area of triangle $\triangle ABC$ in two ways. First, we have $[ABC] = \frac{AB \cdot CE}{2} = 84$. Second, since $\triangle ABC$ is circumscribed

in the circle, we have that $[ABC] = \frac{AB \cdot BC \cdot CA}{4R} = \frac{14 \cdot 13 \cdot 15}{4R}$. This implies that $R = \frac{14 \cdot 13 \cdot 15}{4 \cdot 84} = \frac{65}{8}$.

46. * In the adjacent Figure, we have two Cevians in the triangle ABC , \overline{CE} and \overline{BD} intersecting at F . Knowing that the areas of the triangles DFC , CFB and FBE are equal to 2, 6 and 9 respectively, find the area of the quadrilateral $ADFE$.

- (A) 20 (B) 21 (C) 22
 (D) 23 (E) 24



Answer: D

Let $x = [ADFE]$. We have

$$\frac{[CEA]}{[CEB]} = \frac{2 + x}{15} = \frac{EA}{EB}$$

and

$$\frac{[BDA]}{[BDC]} = \frac{9 + x}{8} = \frac{DA}{DC}.$$

In particular, $\frac{CA}{CD} = \frac{17 + x}{8}$. Also,

$$\frac{[CFD]}{[CFB]} = \frac{2}{6} = \frac{FD}{FB}$$

By Menelaus's Theorem in $\triangle ADB$, for the secant CE , we have

$$\frac{FD}{FB} \cdot \frac{EB}{EA} \cdot \frac{CA}{CD} = 1.$$

This implies that

$$\frac{1}{3} \cdot \frac{15}{2 + x} \cdot \frac{17 + x}{8} = 1,$$

so we get the equation $5(17 + x) = 8(x + 2)$, which implies that $x = 23$.

47. For which value(s) of the real number a does the function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

admit an antiderivative? That is, for which value(s) of a there exist a differentiable function F , such that $F'(x) = f(x)$, for all real numbers x ?

- (A) $(-\infty, \infty)$ (B) $[-1, 1]$ (C) 0
(D) -1 and 1 (E) none

Answer: C

Consider the differentiable function

$$G(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We have that

$$G'(x) = \begin{cases} 2x \cos(1/x) + \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This implies that we have $f(x) = G'(x) - H(x) + K(x)$, where

$$H(x) = \begin{cases} 2x \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad K(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ a & \text{if } x = 0. \end{cases}$$

The function H is continuous so, by the Fundamental Theorem of Calculus it has an antiderivative and clearly G' has an antiderivative, so f admits an antiderivative if and only if the function K admits one. This implies that $a = 0$, since in this case K must be the derivative of a constant function.

48. Let a, b, c be the real roots of $x^3 - 4x^2 - 32x + 17 = 0$. The solution of the equation

$$\sqrt[3]{x-a} + \sqrt[3]{x-b} + \sqrt[3]{x-c} = 0$$

can be written as $\frac{M}{N}$, with the fraction in lowest terms. What is the value of $M + N$?

- (A) 1531 (B) 1532 (C) 1533 (D) 1534 (E) 1535

Answer: A

Let $p = \sqrt[3]{x-a}$, $q = \sqrt[3]{x-b}$, and $r = \sqrt[3]{x-c}$, so $p + q + r = 0$. We have the factorization

$$p^3 + q^3 + r^3 - 3pqr = (p + q + r)(p^2 + q^2 + r^2 - pq - pr - qr).$$

Since $p + q + r = 0$, we get $p^3 + q^3 + r^3 = 3pqr$. This implies that

$$(x-a) + (x-b) + (x-c) = 3\sqrt[3]{(x-a)(x-b)(x-c)}.$$

Since a, b, c are the roots of $x^3 - 4x^2 - 32x + 17 = 0$,

$$(x-a)(x-b)(x-c) = x^3 - 4x^2 - 32x + 17.$$

By Vieta's formulas we have, $a + b + c = 4$, thus we get the equation

$$3x - 4 = 3\sqrt[3]{x^3 - 4x^2 - 32x + 17}.$$

This implies that

$$27x^3 - 108x^2 + 144x - 64 = 27x^3 - 108x^2 - 864x + 459,$$

so $\frac{M}{N} = \frac{523}{1008}$. Therefore, we have $M + N = 1531$.

49. What is the value of the limit

$$\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right] ?$$

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$ (E) $\frac{\pi}{8}$

Answer: A

The function $f : [0, 1] \rightarrow \mathbb{R}$, defined by $f(x) = \frac{1}{\sqrt{4-x^2}}$ is continuous, so it admits a definite integral. Using the definition of a Riemann sum for the interval $[0, 1]$, with intermediary points of the form i/n , we have that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right] = \\ & \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{\sqrt{4 - (\frac{1}{n})^2}} + \frac{1}{\sqrt{4 - (\frac{2}{n})^2}} + \cdots + \frac{1}{\sqrt{4 - (\frac{n}{n})^2}} \right] = \int_0^1 \frac{1}{\sqrt{4-x^2}} dx. \end{aligned}$$

Since $\int \frac{1}{\sqrt{4-x^2}} = \arcsin\left(\frac{x}{2}\right) + C$, we get that

$$\int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \arcsin \frac{1}{2} = \frac{\pi}{6}.$$

50. Two people agree to meet at a coffee shop. They each independently pick a random moment in time between 9 a.m. and 10 a.m. and show up exactly at their selected time. But they are very impatient, and only stay for 5 minutes after when they arrive. What is the probability that they meet?

- (A) $\frac{19}{144}$ (B) $\frac{23}{144}$ (C) $\frac{29}{144}$ (D) $\frac{31}{144}$ (E) $\frac{37}{144}$

Answer: B

Let x and y be the number of minutes past 9 a.m. when the first and the second person arrive. We have that $0 \leq x, y \leq 60$. The pair (x, y) covers every point of the square $[0, 60] \times [0, 60]$. In order for the two people to meet, we must have $|x - y| \leq 5$. This implies that the probability that they meet is the ratio between the area bounded by $|x - y| \leq 5$ (which lies inside the square $[0, 60] \times [0, 60]$) and the area of the square. The area of the region $|x - y| \leq 5$ can be viewed as the area of the square minus the area of two isosceles right triangles with sides 55 and it is equal to $3600 - 2 \cdot \frac{55 \cdot 55}{2} = 575$. Thus, the probability is $575/3600 = 23/144$.