**Thirty-sixth Annual Columbus State Invitational Mathematics Tournament**

Sponsored by

The Columbus State University

Department of Mathematics

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The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

## Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of $+12$ for each correct answer, $-3$ for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of $+200$.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 4, 11, 13, 14, 15, 18, 19, 20, 21, 22, 26, 28, 31, 32, 33, 35, 36, 37, 39, 41, 43, 44, 46, 47, 48, 49, 50.

Throughout the exam, $\overline{AB}$ will denote the line segment from point $A$ to point $B$ and $AB$ will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet**.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1. The average age of three girls is 5 years. If a boy joins the group, then the average age of these four children is 6. What is the boy's age?

(A) 9 (B) 8 (C) 10 (D) 12 (E) 11

2. The lines $2x + 3y - 4 = 0$ and $6x + ay - 3 = 0$ intersect at a right angle. Find the value of $a$.

(A) 2 (B) 4 (C) $-4$ (D) $-1$ (E) 3

3. If $a$ and $b$ are positive real numbers such that $\frac{a}{b} = \frac{\pi}{\sqrt{2}}$ find the value of $\frac{\sqrt[6]{a^9b^5}}{\sqrt[3]{a^3b^4}}$.

(A) $\frac{\pi^2}{\sqrt{2}}$ (B) $\frac{\sqrt{\pi}}{\sqrt[4]{2}}$ (C) $\frac{\pi}{\sqrt{2}}$ (D) $\frac{4}{\pi\sqrt{\pi}}$ (E) $\frac{\pi\sqrt{\pi}}{8}$

4. $*$ If $x^2 + y^2 = 200$ and $xy = 50$, then what is the value of $\left(\frac{x+y}{x-y}\right)^2$?

(A) 50 (B) 4 (C) 3 (D) 200 (E) 12

5. A square with side length $\pi$ is inscribed in a circle. Find the area of the circle.

(A) $\frac{2}{\pi}$ (B) $\pi$ (C) $\frac{\pi^3}{2}$ (D) $\frac{\pi}{2}$ (E) $\frac{\pi^2}{4}$

6. If $f(3x) = \frac{3}{1+x^2}$ for all real numbers $x$ then what is $f(x)$?

(A) $\frac{3}{1+9x^2}$ (B) $\frac{27}{9+x^2}$ (C) $\frac{3x}{1+x^2}$ (D) $\frac{9x}{1+x^2}$ (E) $\frac{x}{9+x^2}$

7. If $\log_x 64 = 3$, then what is $\log_2 x^3$?

(A) 6 (B) 64 (C) 3 (D) 8 (E) 32

8. If the positive numbers $a$ and $b$ satisfy the equations $a^2 + b^2 = 3$ and $a^4 + b^4 = 1$ then find the value of $ab$.

(A) 10 (B) 16 (C) 4 (D) 2 (E) 21

9. What is the product of all solutions of the equation $x^4 - 2x^3 + 4x^2 - 8x + 16 = 0$?

(A) 16 (B) $-8$ (C) 4 (D) $-2$ (E) 1

10. The equation $x^2 - 3x + y^2 + 4y - 12 = 0$ represents a circle. Find the coordinates of its center.

(A) $\left(\frac{3}{2}, 2\right)$ (B) $\left(-\frac{3}{2}, -2\right)$ (C) $\left(-\frac{3}{2}, 2\right)$

(D) $\left(\frac{3}{2}, -2\right)$ (E) $\left(2, \frac{3}{2}\right)$

11. $*$ For what value of $k$ does the system of equations $y = x^2$ and $y = 4x + k$ have a unique solution?

(A) $-3$ (B) 6 (C) 0 (D) 9 (E) $-4$

12. The price of a digital camera was raised by $x\%$ and then after a while the new price was lowered by $y\%$ such that the price of the camera is equal to the price before the raise. To which value is $\dfrac{x-y}{xy}$ equal?

(A) $-0.2$ (B) $-0.01$ (C) 0 (D) 0.01 (E) 0.2

13. $*$ If $A = 7^{8045} - 7^{8044} + 7^{8043} - 7^{8042}$, then what are the last two digits of $A$?

(A) 00 (B) 49 (C) 24 (D) 12 (E) 88

14. ∗ How many zeroes are at the end of 2010!?

(A) 51 (B) 105 (C) 501 (D) 150 (E) 15

15. ∗ What is the number of all positive integer solutions $(x, y)$ of the equation

$$x^2 + y^2 = 2010?$$

(A) 0 (B) 4 (C) 8 (D) 16 (E) 32

16. Boxes A, B, C, D, and E have dimensions as follows

A: 25 by 37 by 4

B: 28 by 35 by 1

C: 23 by 35 by 16

D: 7 by 19 by 40

E: 21 by 7 by 39.

Which box contains the longest segment?

(A) A (B) B (C) C (D) D (E) E

17. If the space shuttle is flying with a speed of 17,321 mi/h at an elevation of 126 mi above the equator and the equatorial radius of the Earth is 3963 mi, then how long (to the nearest minute) does it take to complete one revolution?

(A) 93 minutes (B) 115 minutes (C) 89 minutes

(D) 48 minutes (E) 76 minutes

18. ∗ Find the number of real solutions of the equation $|x - 1| + |x + 1| = 2$.

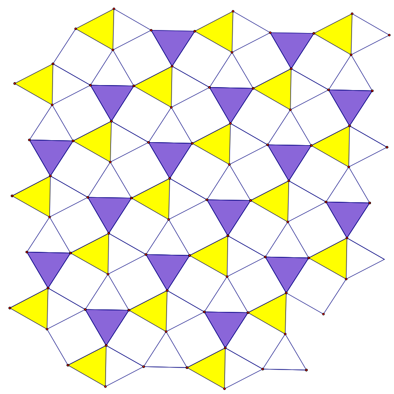
(A) 1 (B) 2 (C) 3

(D) 4 (E) infinite

19. $*$ Let $a, b, c$, and $d$ be positive real numbers such that $\dfrac{2a+b}{2c+d} = \dfrac{3a+b}{3c+d}$ and $\dfrac{4a+b}{4c+d} = 4$. To which of the following is the fraction $\dfrac{5a+b}{5c+d}$ equal?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

20. $*$ Consider the tiling of the plane with squares and equilateral triangles as in the figure. If the pattern continues on a very big area, what is the approximate ratio between the number of squares and the number of equilateral triangles used?



(A) $\dfrac{1}{3}$ (B) 1 (C) $\dfrac{2}{3}$

(D) $\dfrac{1}{2}$ (E) 2

21. $*$ Which is the number of positive divisors of 2010?

(A) 2 (B) 8 (C) 16 (D) 18 (E) 24

22. $*$ How many positive integers are in the set

$$S = \left\{ \frac{2010}{1}, \frac{2011}{2}, \frac{2012}{3}, \frac{2013}{4}, \ldots \right\}?$$

(A) 6 (B) 3 (C) 1 (D) 12 (E) 8

23. How many pairs $(x, y)$ of positive integers with $x + y \leq 120$ satisfy the equation $\dfrac{x^{-1}+y}{x+y^{-1}} = 10$?

(A) 9 (B) 11 (C) 10 (D) 8 (E) 12

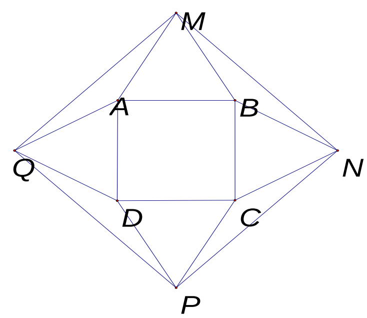
24. Which is the value of the product $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2010}\right)$?

(A) $\frac{2010}{2009}$ (B) $\frac{2011}{2}$ (C) 4020 (D) 1005 (E) $\frac{2009}{2}$

25. Let $\Delta ABC$ be a triangle with sides 3, 4, and 5. What is the sum of the squares of the medians?

(A) $\frac{97}{2}$ (B) $\frac{108}{5}$ (C) $\frac{49}{6}$ (D) $\frac{169}{4}$ (E) $\frac{75}{2}$

26. $*$ Let $ABCD$ be a square. Construct the equilateral triangles $\Delta MAB$, $\Delta NBC$, $\Delta PCD$, and $\Delta QDA$ where $M, N, P, Q$ are exterior to $ABCD$. Find the ratio $\frac{Area(MNPQ)}{Area(ABCD)}$.




(A) $3+\sqrt{2}$ (B) $2+\sqrt{3}$ (C) $\sqrt{2}+\sqrt{3}$

(D) $2\sqrt{2}+\sqrt{3}$ (E) $\sqrt{3}+3\sqrt{2}$

27. A cube is inscribed in a sphere. What is the ratio of the volume of the cube to the volume of the sphere?

(A) $\frac{3\sqrt{2}}{2\pi}$ (B) $\frac{\sqrt{6}}{4\pi}$ (C) $\frac{\sqrt{\pi}}{6}$

(D) $\frac{2\pi\sqrt{3}}{3}$ (E) $\frac{2\sqrt{3}}{3\pi}$

28. $*$ The statement "**If it is snowing then schools are closed**" is logically equivalent to which of the following?

A: "If it is snowing then schools are not closed."

B: "If it is not snowing then schools are closed."

C: “If it is not snowing then schools are not closed.”

D: “If schools are closed then it is snowing.”

E: “If schools are not closed then it is not snowing.”

(A) A (B) B (C) C (D) D (E) E

29. A sphere is inscribed in a cube. What is the ratio of the volume of the sphere to the volume of the cube?

(A) $\pi$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$ (E) $\frac{\pi}{4}$

30. A roulette wheel contains 38 numbers: 18 are red, 18 are black and 2 are green. When the roulette wheel is spun, the ball is equally likely to land on any of the 38 numbers. Suppose that you bet \$1 on red. If the ball lands on a red number, you double your money; otherwise you lose your \$1. In dollars, find with approximation the expected value you will win.

(A) $-0.05$ (B) $-0.03$ (C) 0 (D) 0.03 (E) 0.05

31. $*$ Let $x = 67^{30^{2010}}$, $y = 2010^{67^{30}}$, and $z = 30^{2010^{67}}$. Which of the following is true?

(A) $x < z < y$ (B) $z < x < y$ (C) $z < y < x$

(D) $y < x < z$ (E) $y < z < x$

32. $*$ The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation $f(2x+1) - f(2x-1) = x$ for all real numbers $x$. If $f(1) = 0$, find $f(2011)$.

(A) 1010025 (B) 1015025 (C) 505515

(D) 2001025 (E) 5500125

33. $*$ How many triples $(x, y, z)$ are solutions of the equation $x^2+2y^2+3z^2+\frac{1}{x^2}+\frac{2}{y^2}+\frac{3}{z^2}=12$?

(A) 2 (B) 4 (C) 8 (D) 12 (E) 16

34. Let $0 < b < 1$ and $0 < \alpha < \frac{\pi}{4}$. Define $x = (\sin\alpha)^{\log_b \sin\alpha}$, $y = (\cos\alpha)^{\log_b \cos\alpha}$, and $z = (\sin\alpha)^{\log_b \cos\alpha}$. Which of the following is true?

(A) $y < x < z$ (B) $x < y < z$ (C) $z < x < y$

(D) $y < z < x$ (E) $x < z < y$

35. $*$ Find the sum of the solutions of the equation

$$\sqrt{x+\sqrt{4x-3}+\frac{1}{4}}+\sqrt{x-\sqrt{4x-3}+\frac{1}{4}}=x.$$

(A) 4 (B) 5 (C) 2 (D) 3 (E) 1

36. $*$ If $f(x) = \dfrac{1}{x^2+5x+6}$, find $f(1)+f(2)+\cdots+f(2010)$.

(A) $\dfrac{669}{2011}$ (B) $\dfrac{670}{2012}$ (C) $\dfrac{669}{2012}$ (D) $\dfrac{670}{2013}$ (E) $\dfrac{671}{2013}$

37. $*$ Which is the $2010^{th}$ term of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4,...?

(A) 66 (B) 63 (C) 64 (D) 61 (E) 62

38. An acute triangle $\Delta ABC$ with $m\angle A = 60°$ has sides measurements $\overline{BC} = 2010$ and $\overline{AB} = 2310$. What is the length of the third side?

(A) 1350 (B) 960 (C) 1400 (D) 1050 (E) 85

39. $*$ For how many integers $k$ between 1 and 2010 is the improper fraction $\dfrac{k^2+4}{k+5}$ not in lowest terms?

(A) 70 (B) 68 (C) 67 (D) 66 (E) 69

40. Find the value of $a$ in $[0, 2\pi]$ such that

$$\sin\left(\frac{\pi}{3}+x\right)+\sin(a+x)\geq 0$$

for all real numbers $x$.

(A) $\dfrac{4\pi}{3}$ (B) $\dfrac{3\pi}{4}$ (C) $\dfrac{2\pi}{3}$ (D) $\pi$ (E) $\dfrac{5\pi}{4}$

41. $*$ If $a^x \geq x+1$ for all real numbers $x$ then what is the value of $a$?

(A) 2 (B) $e$ (C) $\pi$ (D) $\pi+e$ (E) $\dfrac{\pi}{e}$

42. In a triangle $\Delta ABC$ we have $BC = 4$, $AC = 3$. Let $\alpha$ be the measure of angle $\angle A$ and $\beta$ be the measure of angle $\angle B$. If $\cos(\alpha-\beta) = \dfrac{3}{4}$ then find $\sin\alpha$.

(A) $\dfrac{3}{5}$ (B) $\dfrac{2}{3}$ (C) $-\dfrac{2}{5}$ (D) $-\dfrac{3}{4}$ (E) 1

43. $*$ Let $f(x)$ be a real valued function defined on the interval $(0, \infty)$. If $f(a) \neq 0$ for some $a$ in $(0, \infty)$ and $f(x^y) = yf(x)$ for all positive real numbers $x$ and $y$, then how many zeroes does $f(x)$ have?

(A) none (B) 1 (C) 2

(D) 3 (E) infinitely many

44. ∗ In a triangle $\Delta ABC$ the segment $\overline{AM}$ is the median of $\overline{BC}$. Let $D$ be the midpoint of $\overline{AM}$ and let $N$ be the point on $\overline{AC}$ such that the points $B, D$, and $N$ are on the same line. Find the ratio $\dfrac{DN}{DB}$.

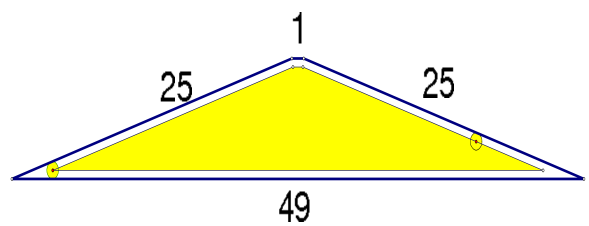
(A) $\frac{2}{5}$ (B) $\frac{3}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{3}{5}$

45. A diagnostic test for a disease is said to be 90% accurate if it detects 90% of the persons who have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability 90%. Only 1% of the population has the disease. A person is selected at random from the population and the diagnostic test indicates that the person has the disease. What is the probability that the person has the disease?

(A) about 92% (B) about 18% (C) about 32%

(D) about 8% (E) about 1%

46. ∗ A billiard table is in the shape of an isosceles trapezoid with dimensions 1, 25, 49, and 25. A ball has diameter 1 and rolls along the inside boundary of the table staying tangent to the side/s at all times. After a full rotation around the table, its center has travel for how long?



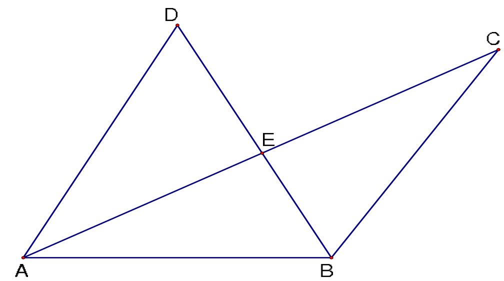

(A) $\frac{900}{7}$ (B) $\frac{400}{7}$ (C) $\frac{500}{7}$

(D) $\frac{600}{7}$ (E) $\frac{800}{7}$

47. ∗ Let $a, b, c$, and $d$ be four positive integers with $a > b > c > d$ such that $a+b+c+d = 2010$ and $a^2 - b^2 + c^2 - d^2 = 2010$. How many distinct values are possible for $a$?

(A) 1005 (B) 501 (C) 2010 (D) 252 (E) 0

48. $\ast$ The triangle $\Delta ADB$ is isosceles (AD=DB=147). The measure of angle $\angle ADB$ is twice the measure of the angle $\angle ACB$. The line segments BD and AC intersect at a point E such that DE=140. What is the product(AE)(EC) ?




(A) 2008 (B) 2009 (C) 2010

(D) 2011 (E) 2012

49. $\ast$ Let $a$ and $b$ be real numbers such that the reminder of the division of $X^{2010} + aX + b$ by $X^2 + X + 1$ is 0. What is the value of $a + b$?

(A) $-1$ (B) 3 (C) 2 (D) 0 (E) 1

50. $\ast$ For each real number $m$ the parabola $y = (m^2 + 4)x^2 + (m - 2)^2 x - 4m + 2$ passes through the same point $(a, b)$. Find the value of $a^2 + b^2$.

(A) 10 (B) 25 (C) 15 (D) 5 (E) 20