

**Solutions of the Thirty-sixth Annual Columbus State Invitational Mathematics
Tournament**

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1 A	6 B	11 E	16 E	21 C	26 B	31 E	36 D	41 B	46 D
2 C	7 A	12 D	17 C	22 A	27 E	32 C	37 B	42 E	47 B
3 B	8 D	13 A	18 E	23 C	28 E	33 C	38 A	43 B	48 B
4 C	9 A	14 C	19 D	24 B	29 B	34 E	39 E	44 C	49 A
5 C	10 D	15 A	20 D	25 E	30 A	35 D	40 A	45 D	50 D

1. The average age of three girls is 5 years. If a boy joins the group, then the average age of these four children is 6. What is the boy's age?

(A) 9 (B) 8 (C) 10 (D) 12 (E) 11

Answer: A

Denote the age of the girls by $x, y,$ and z and the age of the boy by t . Then we have $x + y + z = 15$ and $x + y + z + t = 24$, so $t = 9$.

2. The lines $2x + 3y - 4 = 0$ and $6x + ay - 3 = 0$ intersect at a right angle. Find the value of a .

(A) 2 (B) 4 (C) -4 (D) -1 (E) 3

Answer: C

Note that a can not be 0 and the slopes of the two lines are $-\frac{2}{3}$ and $-\frac{6}{a}$. The lines are perpendicular if and only if the product of their slopes is -1 . It follows that $a = -4$.

3. If a and b are positive real numbers such that $\frac{a}{b} = \frac{\pi}{\sqrt{2}}$ find the value of $\frac{\sqrt[6]{a^9 b^5}}{\sqrt[3]{a^3 b^4}}$.

(A) $\frac{\pi^2}{\sqrt{2}}$ (B) $\frac{\sqrt{\pi}}{\sqrt[4]{2}}$ (C) $\frac{\pi}{\sqrt{2}}$ (D) $\frac{4}{\pi\sqrt{\pi}}$ (E) $\frac{\pi\sqrt{\pi}}{8}$

Answer: B

Using the laws of exponents we get $\frac{\sqrt[6]{a^9b^5}}{\sqrt[3]{a^3b^4}} = \sqrt[6]{\frac{a^9b^5}{a^6b^8}} = \sqrt[6]{\frac{a^3}{b^3}} = \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

4. If $x^2 + y^2 = 200$ and $xy = 50$, then what is the value of $\left(\frac{x+y}{x-y}\right)^2$?

- (A) 50 (B) 4 (C) 3 (D) 200 (E) 12

Answer: C

Use $(x+y)^2 = x^2 + 2xy + y^2$, and $(x-y)^2 = x^2 - 2xy + y^2$. We get $(x+y)^2 = 300$ and $(x-y)^2 = 100$, thus $\left(\frac{x+y}{x-y}\right)^2 = 3$.

5. A square with side length π is inscribed in a circle. Find the area of the circle.

- (A) $\frac{2}{\pi}$ (B) π (C) $\frac{\pi^3}{2}$ (D) $\frac{\pi}{2}$ (E) $\frac{\pi^2}{4}$

Answer: C

Using Pitagora's theorem we find that the radius of the circle is $\frac{\pi\sqrt{2}}{2}$. Now, the formula for the area of a circle gives us $\frac{\pi^3}{2}$.

6. If $f(3x) = \frac{3}{1+x^2}$ for all real numbers x then what is $f(x)$?

- (A) $\frac{3}{1+9x^2}$ (B) $\frac{27}{9+x^2}$ (C) $\frac{3x}{1+x^2}$ (D) $\frac{9x}{1+x^2}$ (E) $\frac{x}{9+x^2}$

Answer: B

Replace x by $\frac{x}{3}$ in the above formula and get $f(x) = \frac{3}{1+(\frac{x}{3})^2}$. After simplifying we get $f(x) = \frac{27}{9+x^2}$.

7. If $\log_x 64 = 3$, then what is $\log_2 x^3$?

- (A) 6 (B) 64 (C) 3 (D) 8 (E) 32

Answer: A

We have $x^3 = 64$, thus $x = 4$. It follows that $\log_2 x^3 = 6$.

8. If the positive numbers a and b satisfy the equations $a^2 + b^2 = 3$ and $a^4 + b^4 = 1$ then find the value of ab .

(A) 10 (B) 16 (C) 4 (D) 2 (E) 21

Answer: D

Use $(a^2 + b^2)^2 = a^4 + 2a^2b^2 + b^4$. We get the equation $9 = 1 + 2a^2b^2$, so $a^2b^2 = 4$. Since a and b are positive we get $ab = 2$.

9. What is the product of all solutions of the equation $x^4 - 2x^3 + 4x^2 - 8x + 16 = 0$?

(A) 16 (B) -8 (C) 4 (D) -2 (E) 1

Answer: A

Let a, b, c , and d the solutions (not necessarily distinct) of the equation. Then $x^4 - 2x^3 + 4x^2 - 8x + 16 = (x - a)(x - b)(x - c)(x - d)$, so the product $abcd = 16$.

10. The equation $x^2 - 3x + y^2 + 4y - 12 = 0$ represents a circle. Find the coordinates of its center.

(A) $\left(\frac{3}{2}, 2\right)$ (B) $\left(-\frac{3}{2}, -2\right)$ (C) $\left(-\frac{3}{2}, 2\right)$
(D) $\left(\frac{3}{2}, -2\right)$ (E) $\left(2, \frac{3}{2}\right)$

Answer: D

Complete the squares to get the equation in the form $\left(x - \frac{3}{2}\right)^2 + (y + 2)^2 = \frac{73}{4}$. Thus

the center of the circle is $\left(\frac{3}{2}, -2\right)$

11. For what value of k does the system of equations $y = x^2$ and $y = 4x + k$ have a unique solution?

(A) -3 (B) 6 (C) 0 (D) 9 (E) -4

Answer: E

The general quadratic equation $ax^2 + bx + c = 0$ admits a unique solution (with multiplicity 2) if and only if the discriminant $b^2 - 4ac = 0$.

The system of equations is equivalent to the equation $x^2 - 4x - k = 0$ which has a unique solution if and only if the discriminant $16 + 4k = 0$. It follows that $k = -4$.

12. The price of a digital camera was raised by $x\%$ and then after a while the new price was lowered by $y\%$ such that the price of the camera is equal to the price before the raise. To which value is $\frac{x-y}{xy}$ equal?

(A) -0.2 (B) -0.01 (C) 0 (D) 0.01 (E) 0.2

Answer: D

Let P be the original price of the camera. The price after the raise is $P\left(1 + \frac{x}{100}\right)$. After lowering the price by $y\%$ the price is $P\left(1 + \frac{x}{100}\right)\left(1 - \frac{y}{100}\right)$ which is equal to the original price P .

Thus we get the equation $P\left(1 + \frac{x}{100}\right)\left(1 - \frac{y}{100}\right) = P$. Eliminate P and multiply to get $\frac{x}{100} - \frac{y}{100} = \frac{xy}{10000}$. It follows that $\frac{x-y}{xy} = 0.01$.

13. If $A = 7^{8045} - 7^{8044} + 7^{8043} - 7^{8042}$, then what are the last two digits of A ?

(A) 00 (B) 49 (C) 24 (D) 12 (E) 88

Answer: A

$A = 7^{8042} \cdot (7^3 - 7^2 + 7 - 1) = 7^{8042} \cdot 300$. Thus the last two digits are 00.

14. How many zeroes are at the end of $2010!$?

(A) 51 (B) 105 (C) 501 (D) 150 (E) 15

Answer: C

Note that each 0 at the end of $2010!$ comes from a factor of 10 of the number $2010!$ and that $k \times 2 \times 5^n = k \times 5^{n-1} \times 10$ for natural numbers k and n . So a natural power of 5 and a factor of 2 produce a factor of 10. There are more even numbers than integer multiples of natural powers of 5 from 1 to 2010. Thus finding the number of zeroes at the end of $2010!$ is equivalent to finding how many integer multiple of the only factors 5, 5^2 , 5^3 and 5^4 smaller than 2010 are there ($5^5 = 3125 > 2010$). Now

$2010 \div 5 = 402$, $2010 \div 25 = 80.4$, $2010 \div 125 = 16.08$, $2010 \div 625 = 3.216$. Therefore, $402+80+16+3=501$.

15. What is the number of all positive integer solutions (x, y) of the equation

$$x^2 + y^2 = 2010?$$

- (A) 0 (B) 4 (C) 8 (D) 16 (E) 32

Answer: A

The equation may be written as $x^2 + y^2 = 2 \cdot 3 \cdot 5 \cdot 67$. A classical result in number theory states that if a prime of the form $4k + 3$ divides the sum of squares $x^2 + y^2$ then it divides both x and y . This means that 3 divides both x and y . If we let $x = 3a$ and $y = 3b$ with a and b integers we get $3(a^2 + b^2) = 670$ which is impossible since 3 is not a divisor of 670.

16. Boxes A, B, C, D, and E have dimensions as follows

A: 25 by 37 by 4

B: 28 by 35 by 1

C: 23 by 35 by 16

D: 7 by 19 by 40

E: 21 by 7 by 39.

Which box contains the longest segment?

- (A) A (B) B (C) C (D) D (E) E

Answer: E

In a box with width w , length l , and height h the longest segment is any diagonal obtained by joining points which do not belong to the same face. Using Pitagora's theorem we get that the length of such a diagonal is $\sqrt{w^2 + l^2 + h^2}$. For our sets of numbers the longest diagonal is $\sqrt{21^2 + 7^2 + 39^2} = \sqrt{2011}$. The other four answers give $\sqrt{2010}$.

17. If the space shuttle is flying with a speed of 17,321 mi/h at an elevation of 126 mi above the equator and the equatorial radius of the Earth is 3963 mi, then how long (to the nearest minute) does it take to complete one revolution?

- (A) 93 minutes (B) 115 minutes (C) 89 minutes
(D) 48 minutes (E) 76 minutes

Answer: C

The length of one revolution is $L = 2\pi R$, where R is the radius of the circle described by the shuttle. Since the radius of the Earth is 3963 miles and the elevation of the shuttle is 126 miles then $R = 3963 + 126 = 4089$ miles. Thus $L = 2\pi 4089 \approx 25692$ miles. The time it takes the shuttle to complete one revolution is approximately $\frac{25692}{17321} \approx 89$ minutes.

18. Find the number of real solutions of the equation $|x - 1| + |x + 1| = 2$.
- (A) 1 (B) 2 (C) 3
- (D) 4 (E) infinite

Answer: E

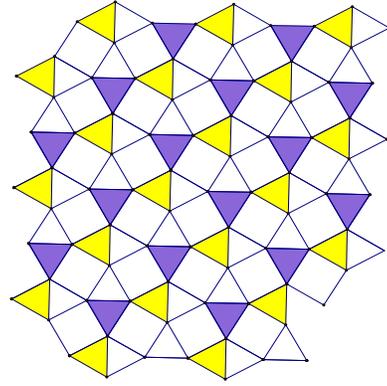
Note that for $-1 < x < 1$ we have $|x - 1| = -x + 1$ and $|x + 1| = x + 1$. Thus for $-1 < x < 1$ the equation becomes $-x + 1 + x + 1 = 2$ which is satisfied for every x . Since there are infinitely real numbers between -1 and 1 the equation has infinitely many solutions.

19. Let a, b, c , and d be positive real numbers such that $\frac{2a + b}{2c + d} = \frac{3a + b}{3c + d}$ and $\frac{4a + b}{4c + d} = 4$. To which of the following is the fraction $\frac{5a + b}{5c + d}$ equal?
- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: D

Cross multiply $\frac{2a + b}{2c + d} = \frac{3a + b}{3c + d}$ to get $ad = bc$. This implies $\frac{a}{c} = \frac{b}{d} = k$, where k is some real number. Since $a = kc$, $b = kd$, and $\frac{4a + b}{4c + d} = 4$ we get $k = 4$. Use $a = 4c$ and $b = 4d$ and get $\frac{5a + b}{5c + d} = 4$.

20. Consider the tiling of the plane with squares and equilateral triangles as in the figure. If the pattern continues on a very big area, what is the approximate ratio between the number of squares and the number of equilateral triangles used?



- (A) $\frac{1}{3}$ (B) 1 (C) $\frac{2}{3}$
 (D) $\frac{1}{2}$ (E) 2

Answer: D

Count the number of equilateral triangles and squares that lie inside a n by n square with the vertices on the sides or the midpoints of the sides of the equilateral triangles (could take n to be the length of a triangle side plus the length of two heights). If a triangle is not entirely inside the n by n square then combine it with another one with the same property so that you form an equilateral triangle. Because the plane can be covered with such squares and the tiling is identical for every such n by n square we get that the quotient is $\frac{1}{2}$.

21. Which is the number of positive divisors of 2010?

- (A) 2 (B) 8 (C) 16 (D) 18 (E) 24

Answer: C

If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ is the decomposition into a product of distinct primes of the positive integer n , then the number of divisors of n is $(1 + \alpha_1)(1 + \alpha_2) \cdots (1 + \alpha_n)$. This is because each factor $p_k^{\alpha_k}$ has $(1 + \alpha_k)$ divisors (they are $1, p_k, p_k^2, \dots, p_k^{\alpha_k}$).

We have $2010 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 67^1$, so the number of divisors of 2010 is $(1 + 1)(1 + 1)(1 + 1)(1 + 1) = 16$.

22. How many positive integers are in the set

$$S = \left\{ \frac{2010}{1}, \frac{2011}{2}, \frac{2012}{3}, \frac{2013}{4}, \dots \right\}?$$

- (A) 6 (B) 3 (C) 1 (D) 12 (E) 8

Answer: A

The set S is generated by the sequence $\frac{2009 + n}{n}$, where n is any positive integer.

An element of S is an integer if and only if $\frac{2009}{n} + 1$ is an integer for some value of n . This implies that n divides $2009 = 7^2 \times 41$. There are six positive divisors of 2009, thus there are six integers in S .

23. How many pairs (x, y) of positive integers with $x + y \leq 120$ satisfy the equation $\frac{x^{-1} + y}{x + y^{-1}} = 10$?

(A) 9 (B) 11 (C) 10 (D) 8 (E) 12

Answer: C

$\frac{x^{-1} + y}{x + y^{-1}} = 10$ implies $\frac{\frac{1+xy}{x}}{\frac{xy+1}{y}} = 10$, so we get $\frac{y}{x} = 10$. Since $0 < x + y \leq 120$ we get $0 < 11x \leq 120$, so $0 < x \leq 10$. Thus there are 10 possibilities for x and consequently 10 pairs (x, y) .

24. Which is the value of the product $\left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \cdots \left(1 + \frac{1}{2010}\right)$?

(A) $\frac{2010}{2009}$ (B) $\frac{2011}{2}$ (C) 4020 (D) 1005 (E) $\frac{2009}{2}$

Answer: B

The product is equal to $\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{2011}{2010} = \frac{2011}{2}$.

25. Let $\triangle ABC$ be a triangle with sides 3, 4, and 5. What is the sum of the squares of the medians?

(A) $\frac{97}{2}$ (B) $\frac{108}{5}$ (C) $\frac{49}{6}$ (D) $\frac{169}{4}$ (E) $\frac{75}{2}$

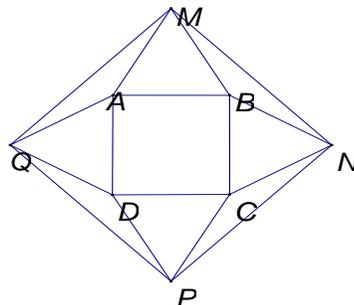
Answer: E

Let \overline{AM} , \overline{BN} , and \overline{CP} be the medians of the triangle $\triangle ABC$. Note that $\triangle ABC$ is a right triangle since $3^2 + 4^2 = 5^2$. Without losing the generality we may say $\angle A = 90^\circ$, and $\overline{AB} = 3$. This implies that $AM = \frac{5}{2}$, so $AM^2 = \frac{25}{4}$. By using Pitagora's theorem in triangle $\triangle ABN$ we get $BN^2 = AB^2 + AN^2 = 9 + 4 = 13$.

Similarly in triangle $\triangle ACP$ we get $CP^2 = AC^2 + AP^2 = 16 + \frac{9}{4} = \frac{73}{4}$. Thus we have

$$AM^2 + BN^2 + CP^2 = \frac{25}{4} + 13 + \frac{73}{4} = \frac{75}{2}$$

26. Let $ABCD$ be a square. Construct the equilateral triangles $\triangle MAB$, $\triangle NBC$, $\triangle PCD$, and $\triangle QDA$ where M, N, P, Q are exterior to $ABCD$. Find the ratio $\frac{\text{Area}(MNPQ)}{\text{Area}(ABCD)}$.



- (A) $3 + \sqrt{2}$ (B) $2 + \sqrt{3}$ (C) $\sqrt{2} + \sqrt{3}$
 (D) $2\sqrt{2} + \sqrt{3}$ (E) $\sqrt{3} + 3\sqrt{2}$

Answer: B

Note that $MNPQ$ is a square since it has all sides of the same length and all angles equal to 90° . Also, note that the angle $\angle QAM = 150^\circ$.

Denote by a the length of a side of the square $ABCD$. Using the Law of Cosines in triangle $\triangle QAM$ we get $QM^2 = AM^2 + AQ^2 - 2AM \cdot AQ \cos 150^\circ = a^2 + a^2 + 2a^2 \frac{\sqrt{3}}{2} = 2a^2 \left(1 + \frac{\sqrt{3}}{2}\right)$. Thus $\frac{\text{Area}(MNPQ)}{\text{Area}(ABCD)} = \frac{QM^2}{AB^2} = \frac{2a^2(1 + \frac{\sqrt{3}}{2})}{a^2} = 2 + \sqrt{3}$

27. A cube is inscribed in a sphere. What is the ratio of the volume of the cube to the volume of the sphere?

- (A) $\frac{3\sqrt{2}}{2\pi}$ (B) $\frac{\sqrt{6}}{4\pi}$ (C) $\frac{\sqrt{\pi}}{6}$
 (D) $\frac{2\pi\sqrt{3}}{3}$ (E) $\frac{2\sqrt{3}}{3\pi}$

Answer: E

Denote by a the length of a side of the cube. The diagonal of the cube has length $a\sqrt{3}$, so the radius R of the sphere is $R = \frac{a\sqrt{3}}{2}$. This implies that the volume of the sphere is $V = \frac{4}{3}\pi R^3 = \frac{\pi\sqrt{3}a^3}{2}$, so the ratio of the volume of the cube to the volume of the

sphere is $\frac{2\sqrt{3}}{3\pi}$.

28. The statement “**If it is snowing then schools are closed**” is logically equivalent to which of the following?

A: “If it is snowing then schools are not closed.”
B: “If it is not snowing then schools are closed.”
C: “If it is not snowing then schools are not closed.”
D: “If schools are closed then it is snowing.”
E: “If schools are not closed then it is not snowing.”

(A) A (B) B (C) C (D) D (E) E

Answer: E

The statement “If schools are not closed then it is not snowing.” is the counterpositive of the statement “If it is snowing then schools are closed” so they are logically equivalent.

29. A sphere is inscribed in a cube. What is the ratio of the volume of the sphere to the volume of the cube?

(A) π (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$ (E) $\frac{\pi}{4}$

Answer: B

Denote by a the length of one side of the cube. Then the radius R of the sphere is equal to $R = \frac{a}{2}$, so the volume of the sphere is $V = \frac{4}{3}\pi R^3 = \frac{\pi a^3}{6}$. This implies that the ratio of the volume of the sphere to the volume of the cube is $\frac{\pi}{6}$.

30. A roulette wheel contains 38 numbers: 18 are red, 18 are black and 2 are green. When the roulette wheel is spun, the ball is equally likely to land on any of the 38 numbers. Suppose that you bet \$1 on red. If the ball lands on a red number, you double your money; otherwise you lose your \$1. In dollars, find with approximation the expected value you will win.

(A) -0.05 (B) -0.03 (C) 0 (D) 0.03 (E) 0.05

Answer: A

The expected value E is equal to $E = \frac{18 \cdot 1 + 20 \cdot (-1)}{38} = -\frac{1}{19} \approx -0.05$.

31. Let $x = 67^{30^{2010}}$, $y = 2010^{67^{30}}$, and $z = 30^{2010^{67}}$. Which of the following is true?

- (A) $x < z < y$ (B) $z < x < y$ (C) $z < y < x$
 (D) $y < x < z$ (E) $y < z < x$

Answer: E

The function “ln” is increasing, so it is enough to compare $\ln(\ln x)$, $\ln(\ln y)$, and $\ln(\ln z)$. We get $\ln(\ln x) = 2010 \ln 30 + \ln(\ln 67) \approx 6837.84$, $\ln(\ln y) = 30 \ln 67 + \ln(\ln 2010) \approx 128.17$, and $\ln(\ln z) = 67 \ln 2010 + \ln(\ln 30) \approx 510.82$. Thus $y < z < x$.

32. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the equation $f(2x + 1) - f(2x - 1) = x$ for all real numbers x . If $f(1) = 0$, find $f(2011)$.

- (A) 1010025 (B) 1015025 (C) 505515
 (D) 2001025 (E) 5500125

Answer: C

Note that we have

$$\begin{cases} f(3) - f(1) = 1 \\ f(5) - f(3) = 2 \\ f(7) - f(5) = 3 \\ \dots \\ \dots \\ f(2011) - f(2009) = 1005 \end{cases}$$

By adding these equations we get $f(2011) = \sum_{i=1}^{1005} i = \frac{1005 \cdot 1006}{2} = 505515$.

33. How many triples (x, y, z) are solutions of the equation $x^2 + 2y^2 + 3z^2 + \frac{1}{x^2} + \frac{2}{y^2} + \frac{3}{z^2} = 12$?

- (A) 2 (B) 4 (C) 8 (D) 12 (E) 16

Answer: C

The equation is equivalent to $\left(x - \frac{1}{x}\right)^2 + 2\left(y - \frac{1}{y}\right)^2 + 3\left(z - \frac{1}{z}\right)^2 = 0$. This implies $x = \frac{1}{x}$, $y = \frac{1}{y}$, and $z = \frac{1}{z}$. Thus $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$. In total there are 8

solutions.

34. Let $0 < b < 1$ and $0 < \alpha < \frac{\pi}{4}$. Define $x = (\sin \alpha)^{\log_b \sin \alpha}$, $y = (\cos \alpha)^{\log_b \cos \alpha}$, and $z = (\sin \alpha)^{\log_b \cos \alpha}$. Which of the following is true?

- (A) $y < x < z$ (B) $x < y < z$ (C) $z < x < y$
 (D) $y < z < x$ (E) $x < z < y$

Answer: E

The logarithmic function “ \log_b ” is decreasing since $0 < b < 1$. We will compare first $\log_b x$, $\log_b y$, and $\log_b z$. Because $0 < \alpha < \frac{\pi}{4}$ we have $\sin \alpha < \cos \alpha$, so $\log_b \sin \alpha > \log_b \cos \alpha$. This implies that $\log_b \cos \alpha \cdot \log_b \cos \alpha < \log_b \cos \alpha \cdot \log_b \sin \alpha < \log_b \sin \alpha \cdot \log_b \sin \alpha$, so $\log_b y < \log_b z < \log_b x$. Thus, since “ \log_b ” is decreasing, we get $x < z < y$.

35. Find the sum of the solutions of the equation

$$\sqrt{x + \sqrt{4x - 3} + \frac{1}{4}} + \sqrt{x - \sqrt{4x - 3} + \frac{1}{4}} = x.$$

- (A) 4 (B) 5 (C) 2 (D) 3 (E) 1

Answer: D

The equation is equivalent to $\sqrt{\left(\frac{\sqrt{4x-3}}{2} + 1\right)^2} + \sqrt{\left(\frac{\sqrt{4x-3}}{2} - 1\right)^2} = x$. So we get

$$\left|\frac{\sqrt{4x-3}}{2} + 1\right| + \left|\frac{\sqrt{4x-3}}{2} - 1\right| = x.$$

Note that the first absolute value term is positive whenever is defined ($x \geq \frac{3}{4}$). For the second we have

$$\left|\frac{\sqrt{4x-3}}{2} - 1\right| = \begin{cases} -\frac{\sqrt{4x-3}}{2} + 1, & \text{if } \frac{3}{4} \leq x < \frac{7}{4} \\ \frac{\sqrt{4x-3}}{2} - 1, & \text{if } x \geq \frac{7}{4} \end{cases}$$

Thus, if $\frac{3}{4} \leq x < \frac{7}{4}$ the equation becomes $\frac{\sqrt{4x-3}}{2} + 1 - \frac{\sqrt{4x-3}}{2} + 1 = x$, so $x = 2$, which is impossible.

If $\frac{7}{4} \leq x$, then the equation becomes $\frac{\sqrt{4x-3}}{2} + 1 + \frac{\sqrt{4x-3}}{2} - 1 = x$, so we need to solve $\sqrt{4x-3} = x$. Squaring both sides we get $4x - 3 = x^2$, with solutions 3 and 1. Since $x \geq \frac{7}{4}$, only $x = 3$ is a solution.

36. If $f(x) = \frac{1}{x^2 + 5x + 6}$, find $f(1) + f(2) + \cdots + f(2010)$.

- (A) $\frac{669}{2011}$ (B) $\frac{670}{2012}$ (C) $\frac{669}{2012}$ (D) $\frac{670}{2013}$ (E) $\frac{671}{2013}$

Answer: D

Note that $\frac{1}{x^2 + 5x + 6} = \frac{1}{x+2} - \frac{1}{x+3}$, so $f(1) + f(2) + \cdots + f(2010) = \sum_{i=1}^{2010} \left(\frac{1}{i+2} - \frac{1}{i+3} \right) = \frac{1}{3} - \frac{1}{2013} = \frac{670}{2013}$

37. Which is the 2010th term of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...?

- (A) 66 (B) 63 (C) 64 (D) 61 (E) 62

Answer: B

Denote by n value of the 2010th term. Then we must have $1 + 2 + 3 + \cdots + (n-1) < 2010 < 1 + 2 + 3 + \cdots + n$. This is because for each positive integer k there are exactly k terms of the sequence equal to k . The above inequalities imply $n = 63$.

38. An acute triangle $\triangle ABC$ with $m\angle A = 60^\circ$ has sides measurements $\overline{BC} = 2010$ and $\overline{AB} = 2310$. What is the length of the third side?

- (A) 1350 (B) 960 (C) 1400 (D) 1050 (E) 85

Answer: A

Using the Law of Cosines we have $BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos 60^\circ$. Thus we get $AC^2 - 2310AC + 1296000 = 0$. Using the quadratic formula we obtain $AC = \frac{2310 \pm \sqrt{152100}}{2}$. There are two possibilities, $AC = 960$, and $AC = 1350$. From the

Law of Sines we have $\frac{\sin \angle A}{BC} = \frac{\sin \angle C}{AB}$, so $\sin \angle C = \frac{77\sqrt{3}}{134}$. This implies $\angle C \approx 84.4^\circ$, so $\angle B \approx 35.6^\circ$. Since $\frac{\sin \angle B}{AC} = \frac{\sin \angle A}{BC}$ we get $AC = 1350$.

39. For how many integers k between 1 and 2010 is the improper fraction $\frac{k^2 + 4}{k + 5}$ not in lowest terms?
- (A) 70 (B) 68 (C) 67 (D) 66 (E) 69

Answer: E

Note that $\frac{k^2 + 4}{k + 5} = k - 5 + \frac{29}{k + 5}$, so the fraction is not in lowest terms if and only if $(k + 5)$ divides 29. This implies $k = 29n - 5$ with n integer. Since k is between 1 and 2010 we must have $1 \leq n \leq 69$, so there are 69 values of k .

40. Find the value of a in $[0, 2\pi]$ such that

$$\sin\left(\frac{\pi}{3} + x\right) + \sin(a + x) \geq 0$$

for all real numbers x .

- (A) $\frac{4\pi}{3}$ (B) $\frac{3\pi}{4}$ (C) $\frac{2\pi}{3}$ (D) π (E) $\frac{5\pi}{4}$

Answer: A

Using the sum to product formula we get

$$\sin\left(\frac{\pi}{3} + x\right) + \sin(a + x) = 2 \sin\left(\frac{\frac{\pi}{3} + a + 2x}{2}\right) \cdot \cos\left(\frac{\frac{\pi}{3} - a}{2}\right)$$

After simplifying we have $2 \sin\left(\frac{\pi}{6} + \frac{a}{2} + x\right) \cdot \cos\left(\frac{\pi}{6} - \frac{a}{2}\right) \geq 0$.

This inequality implies $\cos\left(\frac{\pi}{6} - \frac{a}{2}\right) = 0$ because $\sin\left(\frac{\pi}{6} + \frac{a}{2} + x\right)$ takes both positive and negative values as x varies in \mathbb{R} and $\cos\left(\frac{\pi}{6} - \frac{a}{2}\right)$ does not depend on x .

Thus we have $\frac{\pi}{6} - \frac{a}{2} = \frac{(2k + 1)\pi}{2}$, for some integer k . Since a is between 0 and 2π we obtain $a = \frac{4\pi}{3}$.

41. If $a^x \geq x + 1$ for all real numbers x then what is the value of a ?

- (A) 2 (B) e (C) π (D) $\pi + e$ (E) $\frac{\pi}{e}$

Answer: B

The real variable function $f(x) = a^x - x - 1$ is differentiable, non-negative on \mathbb{R} , and has a minimum at $x = 0$. Since the derivative at a point of minimum is zero, we get $f'(0) = 0$. Since $f'(x) = a^x \ln a - 1$, we have $f'(0) = \ln a - 1$, thus $a = e$. So far this proves that if a number with the above property exists then it must be “e”. To show that “e” satisfies $f(x) = e^x - x - 1 \geq 0$ for all real numbers x , consider $f'(x) = e^x - 1$ and note that the function f is decreasing on $(-\infty, 0)$ and increasing on $[0, \infty)$ thus $f(x) \geq f(0)$, so $e^x - x - 1 \geq 0$.

42. In a triangle $\triangle ABC$ we have $BC = 4$, $AC = 3$. Let α be the measure of angle $\angle A$ and β be the measure of angle $\angle B$. If $\cos(\alpha - \beta) = \frac{3}{4}$ then find $\sin \alpha$.

- (A) $\frac{3}{5}$ (B) $\frac{2}{3}$ (C) $-\frac{2}{5}$ (D) $-\frac{3}{4}$ (E) 1

Answer: E

The Law of Sines implies $\sin \beta = \frac{3}{4} \sin \alpha$, so $\sin \alpha \cdot \sin \beta = \frac{3}{4} \sin^2 \alpha$.

On the other hand we have $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \frac{3}{4}$. We get $\cos \alpha \cos \beta + \frac{3}{4} \sin^2 \alpha = \frac{3}{4}$. It follows that $\cos \alpha \cos \beta = \frac{3}{4} \cos^2 \alpha$. Thus we have two possible cases.

Case 1. $\cos \beta = \frac{3}{4} \cos \alpha$. Since $\sin \beta = \frac{3}{4} \sin \alpha$ we get $1 \sin^2 \beta + \cos^2 \beta = \frac{9}{16} (\sin^2 \alpha + \cos^2 \alpha) = \frac{9}{16}$, so impossible.

Case 2. $\cos \alpha = 0$, so $\alpha = 90^\circ$ and thus $\sin \alpha = 1$.

43. Let $f(x)$ be a real valued function defined on the interval $(0, \infty)$. If $f(a) \neq 0$ for some a in $(0, \infty)$ and $f(x^y) = yf(x)$ for all positive real numbers x and y , then how many zeroes does $f(x)$ have?

- (A) none (B) 1 (C) 2
(D) 3 (E) infinitely many

Answer: B

Note that for $x = 1$, $y = 2$ we get $f(1) = 2f(1)$, so $f(1) = 0$. This implies that $a \neq 1$. Also, we have $f(a^y) = yf(a)$ for every positive real number y . Make the substitution $y = \log_a x$ in the above formula and get $f(x) = \log_a x \cdot f(a)$, so the only solution of $f(x) = 0$ is the solution of $\log_a x = 0$, so $x = 1$.

44. In a triangle $\triangle ABC$ the segment \overline{AM} is the median of \overline{BC} . Let D be the midpoint of \overline{AM} and let N be the point on \overline{AC} such that the points $B, D,$ and N are on the same line. Find the ratio $\frac{DN}{DB}$.

- (A) $\frac{2}{5}$ (B) $\frac{3}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{3}{5}$

Answer: C

Using Menelaus' s theorem in triangle $\triangle AMC$ for the secant BN we get

$\frac{NC}{NA} \cdot \frac{DA}{DM} \cdot \frac{BM}{BC} = 1$. Since $DA = DM$ and $BC = 2BM$ we get $\frac{NC}{NA} = 2$, so $NC = 2NA$. This implies $\frac{NA}{CA} = \frac{1}{3}$. Using Menelus's theorem in tringle $\triangle NBC$ for the secant AM we get $\frac{DN}{DB} \cdot \frac{MB}{MC} \cdot \frac{CA}{NA} = 1$. Since $BM = MC$ we get $\frac{DN}{DB} = \frac{NA}{CA} = \frac{1}{3}$.

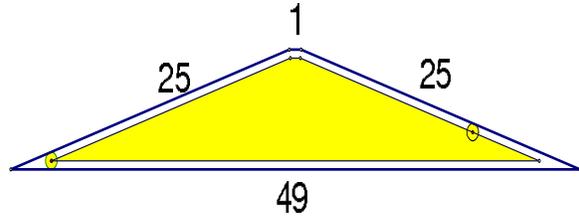
45. A diagnostic test for a disease is said to be 90% accurate if it detects 90% of the persons who have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability 90%. Only 1% of the population has the disease. A person is selected at random from the population and the diagnostic test indicates that the person has the disease. What is the probability that the person has the disease?

- (A) about 92% (B) about 18% (C) about 32%
 (D) about 8% (E) about 1%

Answer: D

Denote by A the event that a person has the disease, by A' the event that a person does not have the disease, and by B the event that a person tests positive for the disease. We want to find the conditional probability $P(A|B)$. By Bayes's theorem we have $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$. Since $P(B|A) = 0.9$, $P(A) = 0.01$ and $P(B) = P(B|A) \cdot P(A) + P(B|A') \cdot P(A') = 0.9 \cdot 0.01 + 0.1 \cdot 0.99 = 0.108$, we get $P(A|B) \approx 8\%$.

46. A billiard table is in the shape of an isosceles trapezoid with dimensions 1, 25, 49, and 25. A ball has diameter 1 and rolls along the inside boundary of the table staying tangent to the side/s at all times. After a full rotation around the table, its center has travel for how long?



- (A) $\frac{900}{7}$ (B) $\frac{400}{7}$ (C) $\frac{500}{7}$
 (D) $\frac{600}{7}$ (E) $\frac{800}{7}$

Answer: D

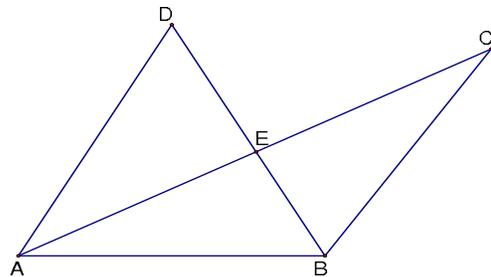
The height of the outer trapezoid is 7 and the height of the inner trapezoid is 6. The triangles formed by the common center of the two trapezoids (center of the inscribed circles) and the vertices are similar, so the proportion is preserved when passing from heights to sides. This implies that the distance traveled is $\frac{6}{7} \cdot (1 + 25 + 49 + 25) = \frac{600}{7}$.

47. Let $a, b, c,$ and d be four positive integers with $a > b > c > d$ such that $a + b + c + d = 2010$ and $a^2 - b^2 + c^2 - d^2 = 2010$. How many distinct values are possible for a ?
- (A) 1005 (B) 501 (C) 2010 (D) 252 (E) 0

Answer: B

We have $a^2 - b^2 + c^2 - d^2 = a + b + c + d$, so $a(a - 1) + c(c - 1) = b(b + 1) + d(d + 1)$. Since $a - 1 \geq b$ and $c - 1 \geq d$ we get that $a - 1 = b$ and $c - 1 = d$. This implies $2a + 2c - 2 = 2010$, so $a + c = 1006$. Since $a > b > c > d$ the largest value for a is 1004 and the smallest is 504, so there are 501 possible values for a .

48. The triangle $\triangle ADB$ is isosceles ($AD = DB = 147$). The measure of angle $\angle ADB$ is twice the measure of the angle $\angle ACB$. The line segments BD and AC intersect at a point E such that $DE = 140$. What is the product $(AE)(EC)$?



- (A) 2008 (B) 2009 (C) 2010
 (D) 2011 (E) 2012

Answer: B

Construct the circle with center D and radius DB and denote with F the symmetric with respect to D of the point B . Because the measure of $\angle ACB$ is half of the measure of $\angle ADB$ the point C lies on the circle. This implies that triangles $\triangle ECB$ and $\triangle EFA$ are similar, so $\frac{EC}{EF} = \frac{EB}{AE}$. It follows that $(AE)(EC) = (EB)(EF) = 7 \cdot 287 = 2009$.

49. Let a and b be real numbers such that the remainder of the division of $X^{2010} + aX + b$ by $X^2 + X + 1$ is 0. What is the value of $a + b$?
- (A) -1 (B) 3 (C) 2 (D) 0 (E) 1

Answer: A

The polynomial $X^2 + X + 1$ has distinct complex, nonreal roots, so it divides $X^{2010} + aX + b$ if and only if its roots are solutions of $X^{2010} + aX + b = 0$. This is equivalent with $r^{2010} + ar + b = 0$ for each root r of $X^2 + X + 1$. If r satisfies $r^2 + r + 1 = 0$, then $r^3 = 1$, since $r^3 - 1 = (r - 1)(r^2 + r + 1)$, so $r^{2010} + ar + b = 0$ is equivalent to $1 + ar + b = 0$. Since a and b are reals and r is a complex but nonreal number we get $b = -1$ and $a = 0$. This means $a + b = -1$.

50. For each real number m the parabola $y = (m^2 + 4)x^2 + (m - 2)^2x - 4m + 2$ passes through the same point (a, b) . Find the value of $a^2 + b^2$.
- (A) 10 (B) 25 (C) 15 (D) 5 (E) 20

Answer: D

Substitute “ a ” in the above expression and write according to the powers of m . We have that the function, in “ m ”, $y = m^2(a^2 + a) - 4m(a + 1) + (4a^2 + 4a + 2)$ is constant (equal to b) for each real number m . This means that $a^2 + a = 0$ and $a + 1 = 0$, so $a = -1$. This implies $b = 2$, so $a^2 + b^2 = 5$.