Thirty-seventh Annual Columbus State Invitational Mathematics Tournament

Sponsored by
The Columbus State University
Department of Mathematics
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The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 3, 5, 7, 9, 12, 13, 15, 17, 19, 20, 21, 23, 27, 28, 30, 31, 34, 36, 38, 40, 41, 44, 45, 47, 48.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet**.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

1. A student writes on the board seven consecutive integers. The sum of the smallest three is 36. Find the sum of the largest three.

   (A) 48 (B) 39 (C) 51 (D) 45 (E) 42

2. Find one quarter of the number 16^{160}.

   (A) 4^{160} (B) 16^{40} (C) 8^{78} (D) 2^{638} (E) 4^{328}

3. * Find the solution of the equation $\dfrac{x-\sqrt{2}}{x-\sqrt{3}} = \dfrac{x-\sqrt{3}}{x-\sqrt{2}}$.

   (A) $\sqrt{3}-\sqrt{2}$ (B) $\dfrac{2}{\sqrt{3}-\sqrt{2}}$ (C) $\dfrac{\sqrt{3}+\sqrt{2}}{2}$

   (D) $\dfrac{2}{\sqrt{3}+\sqrt{2}}$ (E) $\sqrt{3}+\sqrt{2}$

4. The lines $4y+3x-4=0$ and $6y-ax+4=0$ are perpendicular. What is the value of a?

   (A) -2 (B) 8 (C) 6 (D) -4 (E) 0

5. * The expression $\dfrac{4x^{-2}-9y^{-2}}{a-b} \cdot \dfrac{a^{-2}-b^{-2}}{2x^{-1}+3y^{-1}}$ is equivalent to which of the following?

   (A) $\dfrac{(3x+2y)(a-b)}{xyab}$ (B) $\dfrac{(3x-2y)(a-b)}{a^2b^2}$ (C) $\dfrac{(3x+2y)(a+b)}{xy}$

   (D) $\dfrac{(3x+2y)(a+b)}{x^2y^2ab}$ (E) $\dfrac{(3x-2y)(a+b)}{xya^2b^2}$

6. Find all solutions of the equation $\dfrac{1}{x-2}+\dfrac{1}{x+2}=\dfrac{8}{4-x^2}$.

(A) 4 (B) 0 (C) -4

(D) -4 and 4 (E) No solutions

7. $*$ Let a and b be real numbers such that $a+b=5$ and $ab=1$. Which of the following is the quadratic equation with roots a and b?

(A) $5x^2+x+1=0$ (B) $x^2-x+5=0$ (C) $5x^2-x+1=0$

(D) $x^2+5x-1=0$ (E) $x^2-5x+1=0$

8. Find the measure of each interior angle of a regular octagon.

(A) 115° (B) 100° (C) 90° (D) 135° (E) 130°

9. $*$ Let $0<a<b$ be two real numbers such that $a^2+b^2=10ab$. Find the value of $\dfrac{a+b}{a-b}$.

(A) $-\dfrac{\sqrt{6}}{3}$ (B) $-\dfrac{\sqrt{3}}{2}$ (C) $\dfrac{\sqrt{2}}{3}$ (D) $\dfrac{2\sqrt{2}}{3}$ (E) $-\dfrac{\sqrt{6}}{2}$

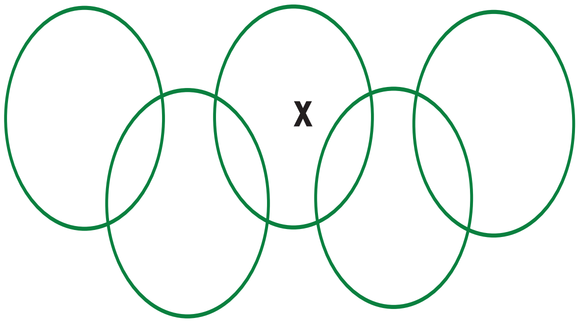
10. Find the sum of the solutions of the equation $(\log_3 x)^2-4\log_3 x+3=0$.

(A) 20 (B) 40 (C) 30 (D) 60 (E) 50

11. For each positive integer n we denote by $s(n)$ the sum of the digits of n. Find the smallest possible value of the ratio $\dfrac{n}{s(n)}$, for $10\le n\le 30$.

(A) 1.7 (B) 1.9 (C) 2 (D) 1.8 (E) 1.6

12. $*$ The five intersecting circles in the figure determine nine regions. In each of these regions you write one of the numbers from 1 to 9 such that each number is written exactly once and the sum of the numbers inside each circle is 11. Which number must be written in the region marked with an **X**?



(A) 7 (B) 4 (C) 6

(D) 3 (E) 5

13. $*$ The sum of 2011 consecutive integers is 2011. Find the sum of the digits of the largest of these integers.

(A) 5 (B) 6 (C) 4 (D) 8 (E) 7

14. Let x be a real number such that $x + \dfrac{1}{x} = 4$. Find the value of $x^3 + \dfrac{1}{x^3}$.

(A) 40 (B) 44 (C) 48 (D) 52 (E) 56

15. $*$ Find the sum $\dfrac{\sqrt{2}-\sqrt{1}}{\sqrt{2}} + \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{6}} + \cdots + \dfrac{\sqrt{2011}-\sqrt{2010}}{\sqrt{2010 \cdot 2011}}$.

(A) $\dfrac{\sqrt{2010}}{\sqrt{2011}}$ (B) $\dfrac{1}{\sqrt{2011}}$ (C) $\dfrac{\sqrt{2011}-1}{\sqrt{2011}}$

(D) $\sqrt{2011}$ (E) $\dfrac{\sqrt{2011}-\sqrt{2010}}{\sqrt{2011}}$

16. Find the number of elements of the set $A = \{2, 3, 6, 7, 10, 11, 14, 15, \ldots, 2006, 2007, 2010, 2011\}$.

(A) 1004 (B) 1006 (C) 1008 (D) 1010 (E) 1012

17. $*$ The positive integers x and y satisfy the relation $3x = 7y$. Which of the following is a possible value for $x + y$?

(A) 2010 (B) 1005 (C) 201 (D) 335 (E) 402

18. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ and m and n real numbers such that

$$mf(x-1) + nf(-x) = 2|x| + 1$$

for all real numbers x. If $f(-2) = 5$ and $f(1) = 1$, then find the numbers m and n.

(A) $m = \frac{1}{6}; n = \frac{5}{6}$ (B) $m = \frac{3}{4}; n = \frac{7}{4}$ (C) $m = \frac{3}{5}; n = \frac{8}{5}$

(D) $m = \frac{5}{12}; n = \frac{11}{12}$ (E) $m = \frac{2}{5}; n = 1$

19. $*$ Find the quadratic function $f(x) = ax^2 + bx + c$ knowing that for every positive integer n we have $f(1) + f(2) + \cdots + f(n) = n^3$.

(A) $f(x) = x^2 - 4x + 4$ (B) $f(x) = 4x^2 + 2x - 3$ (C) $f(x) = 5x^2 - 3x + 2$

(D) $f(x) = 3x^2 - 3x + 1$ (E) $f(x) = x^2 + 3x - 3$

20. $*$ How many distinct permutations of the letters of the word "**PEPPER**" are there?

(A) 720 (B) 40 (C) 360 (D) 120 (E) 60

21. $*$ Find the number of positive integers n, $1 \leq n \leq 100$, such that n^n is a perfect square.

(A) 53 (B) 52 (C) 50 (D) 55 (E) 51

22. For n positive integer, let $S_n = \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n+1}+\sqrt{n}}$.
Find the smallest value of n such that $S_n \geq 100$.

(A) 10,200 (B) 10,000 (C) 9,400 (D) 9,600 (E) 9,800

23. $*$ Find the number of pairs of integers (x, y) which satisfy the equation

$$\frac{1}{x+2010} + \frac{2011}{y+2011} = 1.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

24. Which is the coefficient of x^{35} in the binomial expansion of $(1+x^5)^{10}$?

(A) 120 (B) 210 (C) 250 (D) 320 (E) 370

25. How many pairs (x, y) of positive integers satisfy $x^2 + y^2 = 2011$?

(A) 2011 (B) 2 (C) 0 (D) 1005 (E) 10

26. The number 2011 is written as the sum of $(n+1)$ consecutive integers

$$2011 = a + (a+1) + (a+2) + \cdots + (a+n).$$

If $1 < n < 2011$, then find the value of a.

(A) -62 (B) -250 (C) -125 (D) -502 (E) -1004

27. $*$ Find the number of integers n such that $\frac{n^2+2}{n+1}$ is an integer.

(A) 4 (B) 0 (C) 6 (D) 2 (E) 8

28. $*$ Find the integers a and b such that $\frac{a}{\sqrt{3+2\sqrt{2}}} + \frac{b}{\sqrt{3-2\sqrt{2}}} = \sqrt{11+6\sqrt{2}}.$

(A) $a = 2$, $b = -1$ (B) $a = -2$, $b = 1$ (C) $a = -1$, $b = 2$

(D) $a = 1$, $b = -2$ (E) $a = 1$, $b = 2$

29. Find the values of the real number a such that the equations $x^2 + x + a = 0$ and $ax^2 + x + 1 = 0$ have exactly one common root.

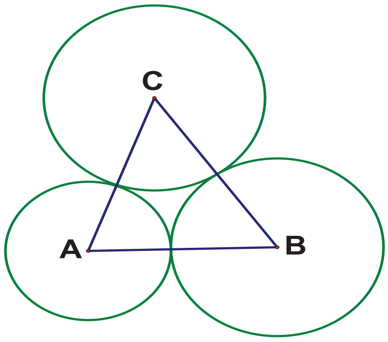
(A) -1 and 0 (B) -1 and 2 (C) 0 and 2

(D) -2 and 2 (E) -2 and 0

30. $*$ Three circles with radii 6, 7, and 8 and centers A, B, and C are tangent to each other as in the figure. Find the area of the triangle ΔABC.

(A) 48 (B) 60 (C) 72

(D) 84 (E) 96



31. $*$ Solve the inequality $\sin x + \sin 2x \geq 0$, for x in $[0, 2\pi)$.

(A) $\left[0, \frac{2\pi}{3}\right] \cup \left[\pi, \frac{4\pi}{3}\right]$ (B) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{4\pi}{3}, 2\pi\right)$ (C) $\left[0, \frac{7\pi}{12}\right] \cup \left[\pi, \frac{17\pi}{12}\right]$

(D) $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$ (E) $[0, \pi] \cup \left[\frac{5\pi}{3}, 2\pi\right)$

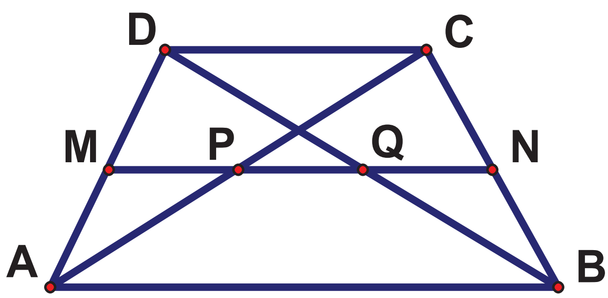
32. Let a, b, and c be positive real numbers such that $\frac{2a}{b+4c} = \frac{b}{2a+4c} = \frac{4c}{2a+b}$. Find the value of $\left(\frac{1}{3a} + \frac{1}{b} + \frac{1}{4c}\right)(3a + b + 4c)$.

(A) $\frac{28}{3}$ (B) $\frac{35}{3}$ (C) $\frac{49}{3}$ (D) $\frac{56}{3}$ (E) $\frac{63}{3}$

33. Find the number of all positive integers n such that $N = (n^2 + 6)^2 - (4n + 2)^2$ is a prime number.

(A) 0 (B) 6 (C) 10 (D) 2 (E) 8

34. $*$ Let $ABCD$ be an isosceles trapezoid such that $\overline{AB} \parallel \overline{CD}$, $AD = BC = DC = 18$, and $m\angle A = 60°$. Let M and N be the midpoints of $\overline{AD}$ and $\overline{BC}$. The segment $\overline{MN}$ intersects the diagonal $\overline{AC}$ at P and the diagonal $\overline{BD}$ at Q. Find the length of the segment $\overline{PQ}$.



(A) 7 (B) 9 (C) 8

(D) 10 (E) 6

35. Let ΔABC be a triangle with sides $AB = 10$, $AC = 24$, and $BC = 26$. Find the length of the median corresponding to $\overline{BC}$.

(A) 12 (B) 13 (C) 5 (D) 17 (E) 7

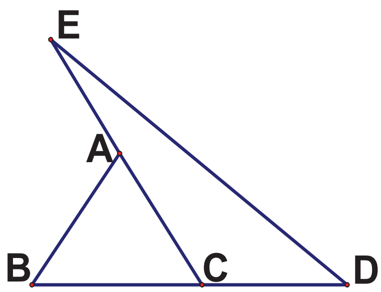
36. $*$ Three balls are randomly drawn, without replacement, from a bowl containing 6 yellow and 5 blue balls. Find the probability that one of the balls is yellow and the other two are blue.

(A) $\frac{4}{11}$ (B) $\frac{3}{11}$ (C) $\frac{7}{11}$ (D) $\frac{5}{11}$ (E) $\frac{6}{11}$

37. Three constructors built 100 houses in a new subdivision. Constructor A built 20 houses, B built 30, and C built 50. The probabilities of developing a leaky basement are 10%, 8%, and 3% for constructors A, B, and C, respectively. Suppose that one basement is leaking. Find the probability that the house was built by constructor A.

(A) 0.059 (B) 0.041 (C) 0.41 (D) 0.34 (E) 0.25

38. ∗ An equilateral triangle ΔABC has sides equal to 10. Let D be a point on the line passing through B and C such that $BC{=}CD$, and let E be a point on the line passing through C and A such that $CA{=}AE$. Find the length of the segment ED.



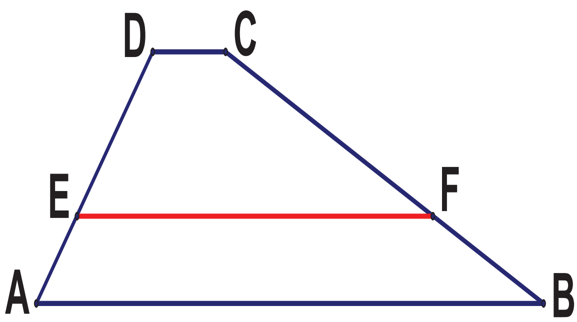
(A) $11\sqrt{6}$ (B) $12\sqrt{5}$ (C) $10\sqrt{7}$

(D) $8\sqrt{10}$ (E) $7\sqrt{13}$

39. If two fair dice are tossed, then find the least number of throws for which the probability of getting at least one double five exceeds 0.5?

(A) 27 (B) 14 (C) 25 (D) 23 (E) 19

40. ∗ The trapezoid $ABCD$ has bases $AB = 2814$ and $CD = 402$. The segment $\overline{EF}$ is parallel to the bases and divides $ABCD$ in two regions with equal area. Find the length of $\overline{EF}$.



(A) 1608 (B) 2010 (C) 2412

(D) 804 (E) 1206

41. ∗ Let a, b, c, and d be positive real numbers such that $ac = bd = 10$. Find the smallest possible value of the expression $E = (a + 2)(b + 2)(c + 5)(d + 5)$.

(A) 1000 (B) 1200 (C) 1400 (D) 1600 (E) 1800

42. Assume $x > 0$. Which is the correct formula for $\arctan(x) + \arctan\left(\frac{1}{2x}\right)$?

(A) $\arctan\left(x + \frac{1}{x}\right)$ (B) $\arctan\left(x + \frac{1}{2x}\right)$ (C) $\arctan\left(2x + \frac{1}{2x}\right)$

(D) $\arctan\left(x + \frac{2}{x}\right)$ (E) $\arctan\left(2x + \frac{1}{x}\right)$

43. Find the positive real number a such that $x^a \leq a^x$, for all positive real numbers x.

(A) e (B) π (C) $\sqrt{e}$ (D) $\pi - \frac{1}{2}$ (E) $e + \frac{1}{2}$

44. $*$ Let $S = 1^{2011} + 2^{2011} + 3^{2011} + \cdots + 2010^{2011}$. Find the remainder of the division of S by 2011.

(A) 2010 (B) 0 (C) 2009 (D) 1 (E) 1005

45. $*$ Let be a function defined for all points $P(x, y)$ in the plane by the formula $f(P) = 23x - 27y$. If $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_{503}(x_{503}, y_{503})$ are the vertices of a regular polygon centered at the point $(26, 22)$, then find the value of $f(P_1) + f(P_2) + \ldots.. + f(P_{503})$.

(A) 8048 (B) 1006 (C) 6036 (D) 4024 (E) 2012

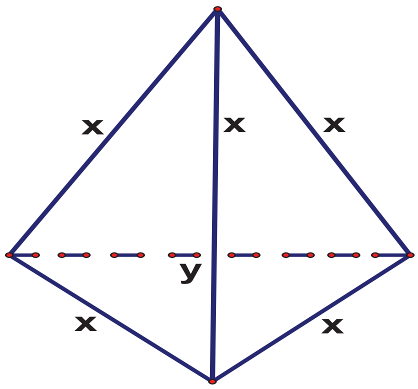
46. Let x, y, and z be three positive real numbers such that $x + y + z = \sqrt{10 + \sqrt{19}}$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \sqrt{10 - \sqrt{19}}$. If $a = x^2 + y^2 + z^2$ and $b = a + \frac{9}{a}$, then find the value of b.

(A) 7 (B) $\frac{20}{3}$ (C) $\frac{22}{3}$ (D) $\frac{19}{3}$ (E) $\frac{23}{3}$

47. $*$ Let α be the acute angle formed by the tangent lines to the graphs of $y = \sin x$ and $y = \cos x$ at $x = \frac{\pi}{4}$. Find $\cos \alpha$.

(A) $\frac{2}{5}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{3}{5}$ (E) $\frac{1}{3}$

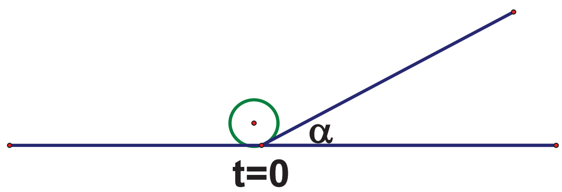
48. $*$ One picks at random two numbers in the interval $[0,1]$, first x and then y. What is the probability that a tetrahedron with side lengths as in the figure can be constructed?



(A) $1-\dfrac{\sqrt{2}}{2}$ (B) $1-\dfrac{\sqrt{3}}{6}$ (C) $1-\dfrac{\sqrt{6}}{3}$

(D) $1-\dfrac{\sqrt{6}}{4}$ (E) $1-\dfrac{\sqrt{2}}{8}$

49. An object is projected with an initial velocity v_0 up a slope that makes an angle α with the horizontal. Assume frictionless motion and that $t=0$ is when the object enters the slope. If g denotes Earth's gravity, then find the time required for the object to return to its initial position at $t=0$.



(A) $\dfrac{2v_0}{g\sin\alpha}$ (B) $\dfrac{v_0}{2g\sin\alpha}$ (C) $\dfrac{v_0}{g\sin\alpha}$

(D) $\dfrac{v_0}{g\cos\alpha}$ (E) $\dfrac{2v_0}{g\cos\alpha}$

50. Alin and Carlos are in empty space (where there is no gravity and their gravitational attraction is so small that it can be considered to be zero) separated by a distance $\mathbf{d}$. Alin throws a ball that reaches Carlos at time $\mathbf{T}$. Denote by $\mathbf{m_A}, \mathbf{m_C}$, and $\mathbf{m_B}$ the masses of Alin, Carlos and the ball respectively. Find Alin's speed after the ball is thrown.

(A) $\dfrac{\mathbf{m_A T}}{\mathbf{m_B d}}$ (B) $\dfrac{\mathbf{m_B d}}{\mathbf{m_A T}}$ (C) $\dfrac{\mathbf{m_B d}}{(\mathbf{m_A}+\mathbf{m_C})\mathbf{T}}$

(D) $\dfrac{(\mathbf{m_A}+\mathbf{m_C})\mathbf{T}}{\mathbf{m_B d}}$ (E) $\mathbf{0}$