Fortieth Annual Columbus State Invitational Mathematics Tournament

Sponsored by

The Columbus State University
Department of Mathematics
March 1, 2014

****

The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We wish you success on this test and in your future studies.

Instructions

This is a 90-minute, 50-problem, multiple choice examination. There are five possible responses to each question. You should select the one "*best*" answer for each problem. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer circle the choice you have made on the test booklet. Carefully transfer your answers to the score sheet. Completely darken the blank corresponding to the letter of your response to each question. Mark your answer boldly with a No. 2 pencil. If you must change an answer, completely erase the previous choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, −3 for each incorrect selection, and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 3, 5, 7, 9, 12, 13, 15, 17, 19, 20, 21, 23, 27, 28, 30, 31, 34, 36, 38, 40, 41, 44, 45, 47, 48.

Throughout the exam, $\overline{AB}$ will denote the line segment from point A to point B and AB will denote the length of $\overline{AB}$. Pre-drawn geometric figures are not necessarily drawn to scale. The measure of the angle $\angle ABC$ is denoted by $m\angle ABC$.

Review and check your score sheet carefully. **Your student identification number and your school number must be encoded correctly on your score sheet**.

When you complete your test, bring your pencil, scratch paper and answer sheet to the test monitor. Leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Do not congregate outside the doors by the testing area. You may keep your copy of the test. Your sponsor will have a copy of solutions to the test problems.

Do not open your test until instructed to do so!

I

1. How many positive integer divisors does 2014 have?

    (A) 2   (B) 4   (C) 6   (D) 8   (E) 10

2. Which number is three times the product of its two digits?

    (A) 12   (B) 18   (C) 24   (D) 26   (E) 32

3. * If the arithmetic mean of two numbers is $10a$ and one of the numbers is $4a+5$, what is the other number?

    (A) $16a-5$ (B) $15a-15$ (C) $14a-25$ (D) $13a-15$ (E) $12a-5$

4. If x is a real number then what is the minimum possible value of $(2x-5)^2 + 18$?

    (A) 18   (B) 16   (C) 14   (D) 12   (E) 10

5. * Which answer below is equal to $\left(\sqrt{5}^{\sqrt{5}}\right)^{\sqrt{5}}$?

    (A) $5\sqrt{5}$ (B) $75\sqrt{5}$ (C) $125\sqrt{5}$ (D) $50\sqrt{5}$ (E) $25\sqrt{5}$

6. For all positive real numbers a and b we define $a \star b = \dfrac{a}{b} + \dfrac{b}{a}$. Determine the value of $2 \star 5$.

    (A) $\dfrac{21}{10}$ (B) $\dfrac{27}{10}$ (C) $\dfrac{29}{10}$ (D) $\dfrac{33}{10}$ (E) $\dfrac{37}{10}$

7. * How many integers between 100 and 1000 are multiples of 11?

    (A) 79   (B) 80   (C) 81   (D) 82   (E) 83

II

8. Let a and b be real numbers such that $a + b = 7$ and $ab = 2$. Which of the following is the quadratic equation with roots a and b?

    (A) $x^2 - 2x + 7 = 0$ (B) $2x^2 - x + 7 = 0$ (C) $x^2 + 2x - 7 = 0$

    (D) $2x^2 - 7x + 1 = 0$ (E) $x^2 - 7x + 2 = 0$

9. $*$ The number n is the biggest positive integer such that $4n$ is a three digit number. The number m is the smallest positive integer such that $3m$ is a three digit number. Find the number $n - m$.

    (A) 215   (B) 217   (C) 219   (D) 221   (E) 223

10. Find the sum of the solutions of the equation $\dfrac{1}{n-7} - \dfrac{1}{n} = \dfrac{1}{14}$.

    (A) 0   (B) 7   (C) 14   (D) 21   (E) 28

11. How many pairs of two-digit positive integers (a, b) satisfy $a - b = 49$?

    (A) 40   (B) 41   (C) 49   (D) 50   (E) 51

12. $*$ When a certain solid substance is melted its volume increases by $\dfrac{1}{8}$. By how much does the volume decrease when the substance solidifies?

    (A) $\dfrac{1}{7}$ (B) $\dfrac{1}{8}$ (C) $\dfrac{1}{9}$ (D) $\dfrac{1}{10}$ (E) $\dfrac{1}{11}$

13. $*$ The real numbers x and y satisfy the system of equations $xy - x = 25$ and $xy + y = 36$. Find $x - y$.

    (A) -5 (B) -1 (C) 0   (D) 1   (E) 5

14. Which digit is in the ones' place of $8^{2013} + 9^{2014}$?

(A) 1   (B) 3   (C) 5   (D) 7   (E) 9

15. * In the adjacent figure we have 18 cubes of side-lengths 1 meter stacked as shown. A smart ant is traveling on the surface of theses cubes from point A to point B. What is the shortest distance (in meters) the ant can take?

(A) 8   (B) 7   (C) 6

(D) 5   (E) 4

16. Let a, b be real numbers such that $\dfrac{5a + 3b}{3a + 5b} = \dfrac{11}{13}$. Find $\dfrac{7a + 4b}{4a + 7b}$.

(A) $\dfrac{5}{6}$ (B) $\dfrac{4}{5}$ (C) $\dfrac{3}{4}$ (D) $\dfrac{2}{3}$ (E) $\dfrac{1}{2}$

17. * If x is a real number such that $|2014 - x| + \sqrt{x - 2015} = x$, then what is the value of $x - 2014^2$?

(A) 2011   (B) 2012   (C) 2013   (D) 2014   (E) 2015

18. Find the number of pairs of real numbers (a, b) which satisfy the equation $a^2 + b^2 = 2a + 2b$.

(A) 1   (B) 2   (C) 4

(D) 8   (E) infinitely many

19. * The perimeter of a right triangle is 16 ft. The sum of the squares of its sides is 98 ft. Find the area of the triangle.

(A) 1 ft^2 (B) 8 ft^2 (C) 16 ft^2 (D) 24 ft^2 (E) 32 ft^2

20. * The polynomial $19X^{2014} - 53X^{1007} + 2X^2 + 2014$ has 2014 complex roots. Find the product of all these roots.

(A) 19   (B) 38   (C) 53   (D) 106   (E) 152

21. * Find the number of nonnegative integers n such that $\dfrac{n^2}{n+6}$ is an integer.

(A) 3   (B) 4   (C) 5   (D) 6   (E) 7

22. Find the sum of the integers a and b such that

$$\frac{a}{\sqrt{4+2\sqrt{3}}} + \frac{b}{\sqrt{4-2\sqrt{3}}} = \sqrt{7+4\sqrt{3}}.$$

(A) 2   (B) 4   (C) 6   (D) 8   (E) 10

23. * Find the number of solutions of the equation $\dfrac{\log_2(x-3)}{\log_2(x^2-3)} = \dfrac{1}{2}$.

(A) 0   (B) 1   (C) 2   (D) 3   (E) 4

24. Find the number of solutions of the equation $2^{x+3} + 2^{x+2} + 2^{x+1} = 7^x + 7^{x-1}$.

(A) 0   (B) 1   (C) 2   (D) 3   (E) 4

25. Let $ABCD$ be a trapezoid such that $\overline{AB} \parallel \overline{CD}$, $\overline{AC} \perp \overline{BD}$, $AC = 65$ cm, and $BD = 48$ cm. Find the area of $ABCD$.

(A) 195 cm²   (B) 390 cm²   (C) 780 cm²

(D) 1560 cm²   (E) 3120 cm²
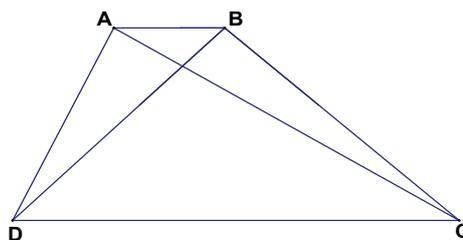
26. Let $S_n = \dfrac{1}{\sqrt{2}+1} + \dfrac{1}{\sqrt{3}+\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$. Determine the smallest integer n such that $S_n > 2014$.

   (A) 2012^2 (B) 2013^2 (C) 2014^2 (D) 2015^2 (E) 2016^2

27. $*$ Consider the function $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$ defined for all $x \in (-1, 1)$. Also, let the function g defined on $(-1, 1)$ by $g(x) = \dfrac{3x + x^3}{1 + 3x^2}$. Then the composition function $f \circ g$ can be simplified to which of the following?

   (A) $3f$ (B) $2f$ (C) f (D) g (E) $2g$

28. $*$ Let x be a real number such that $\sin x - \cos x = \dfrac{1}{2}$. Find the value of $\sin 2x$.

   (A) $\dfrac{3}{8}$ (B) $\dfrac{1}{4}$ (C) $\dfrac{3}{4}$ (D) $\dfrac{5}{8}$ (E) $\dfrac{1}{2}$

29. Let $\triangle ABC$ be a triangle with sides $AB = 5$ cm, $AC = 12$ cm, and $BC = 13$ cm. Find the length of the median corresponding to $\overline{BC}$.

   (A) $\dfrac{7}{2}$ cm   (B) $\dfrac{9}{2}$ cm   (C) $\dfrac{11}{2}$ cm   (D) $\dfrac{13}{2}$ cm   (E) $\dfrac{15}{2}$ cm

30. $*$ Find the number of integers n such that $\sqrt{n^2 - 24}$ is an integer.

   (A) 2   (B) 4   (C) 6   (D) 8   (E) 10

31. $*$ Find how many integers k, $1 \le k \le 50$, have the property that

$$1 + 2 + \cdots + 49 + 50 - k$$

is a perfect square.

   (A) 0   (B) 1   (C) 2   (D) 3   (E) 4

32. In the adjacent figure we have a triangle T (shaded) with vertices $(1,0), (0,1)$, and $(2,2)$, and three of its dilations: $2T, 3T$, and $4T$. One can count the lattice points (integer coordinates) inside of the triangles. We denote by $I(n)$ the number of points of the lattice inside the dilation nT. For instance, $I(1) = 1$, $I(2) = 4$, and $I(3) = 10$. Find $I(9)$.

(A) 105   (B) 106   (C) 107

(D) 108   (E) 109
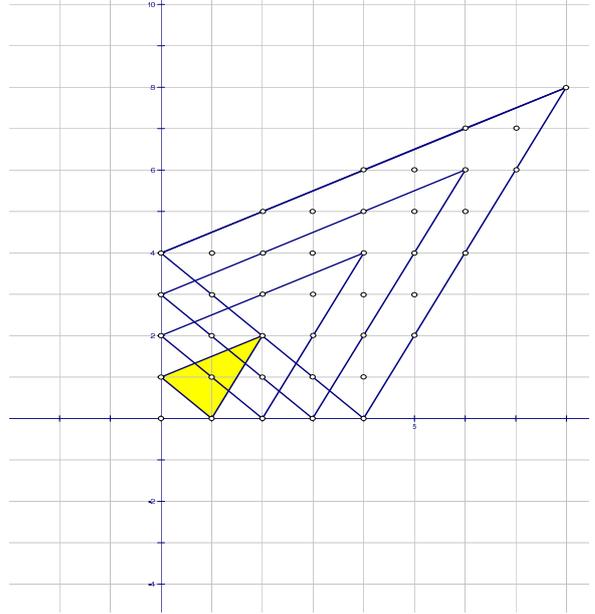
33. Let $ABCD$ be a parallelogram and let M be a point on the segment $\overline{AB}$ such that the area of $\triangle MBC = 8$ cm^2 and the area of $\triangle MDC = 20$ cm^2. Find the area of the triangle $\triangle MAD$.

(A) 4 cm^2 (B) 6 cm^2 (C) 8 cm^2

(D) 10 cm^2 (E) 12 cm^2
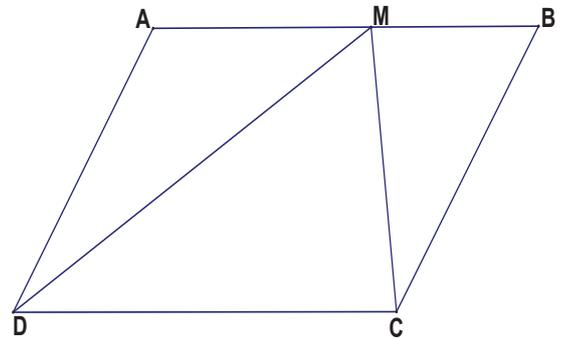
34. ∗ It is known that every positive integer can be written as the sum of non-consecutive Fibonacci numbers F_n in a unique way. Taking into consideration this writing for 2014, let

$$2014 = F_{n_1} + F_{n_2} + \cdots + F_{n_k}.$$

Find the number k.

(The Fibonacci numbers F_n are defined as follows: $F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}$, for $n \geq 2$.)

(A) 1   (B) 2   (C) 3   (D) 4   (E) 5

35. Find the product of the solutions of the equation $(4 + \sqrt{15})^x + (4 - \sqrt{15})^x = 62$.

(A) -25 (B) -16 (C) -9 (D) -4 (E) -1

36. ∗ Find how many positive integers n have the property that $\sqrt{(n!)^2 + 13}$ is an integer, where $n! = 1 \cdot 2 \cdot 3 \cdots \cdot n$.

(A) 0   (B) 1   (C) 2

(D) 3   (E) infinitely many

37. Find the smallest positive integer n such that we can express 2014 as

$$2014 = \pm 1 \pm 2 \pm \cdots \pm n.$$

for some choice of the signs $\pm$. (For example, $8 = -1 + 2 + 3 + 4$ and the smallest n such that 8 can be written as $\pm 1 \pm 2 \cdots \pm n$ for some choice of the signs $\pm$ is $n = 4$.)

(A) 60   (B) 61   (C) 62   (D) 63   (E) 64

38. ∗ The sum of the first n positive integers is equal to a three digit number which has all digits equal. Find the sum of the digits of n.

(A) 9   (B) 13   (C) 17   (D) 21   (E) 25

39. Let a and b be two integers such that the polynomial $f = X^4 - 2X^3 + aX^2 + bX + 1$ has the root $\dfrac{3 - \sqrt{5}}{2}$. Find the sum $a + b$.

(A) -6 (B) -5 (C) -4 (D) -3 (E) -2

40. ∗ Find the number of ordered triples of positive integers (x, y, z) with $x < y < z \leq 2014$ and $x^2 + y^2 + z^2 = xy + yz + zx + 3$.

(A) 2011   (B) 2012   (C) 2013   (D) 2014   (E) 2015

41. * The number 2^{1230} is written after the number 3^{3450} to form a new number. How many digits does the new number have?

(A) 2014   (B) 2015   (C) 2016   (D) 2017   (E) 2018

42. Let a be a positive integer such that the graph of the quadratic function $y = ax^2 + bx + c$ passes through the points $(-1, 4)$ and $(2, 1)$ and has two intersections with the x-axis. What is the maximum value of $b + c$?

(A) -1 (B) -2 (C) -3 (D) -4 (E) -5

43. Consider the number

$$M = (3^{2^0} + 1)(3^{2^1} + 1)(3^{2^2} + 1) \cdots (3^{2^{20}} + 1).$$

Which are the last two digits (base 10) of M?

(A) 20   (B) 22   (C) 24   (D) 26   (E) 28

44. * In the accompanying figure we have an equilateral triangle ABC and the points D, E, and F are on the sides $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$ respectively. Knowing that

$$\frac{AC}{AD} = \frac{AB}{EB} = \frac{BC}{CF} = 3,$$

find the ratio of the area of $\triangle ABC$ to the area of $\triangle DEF$.

(A) 1   (B) 2   (C) 3

(D) 4   (E) 5
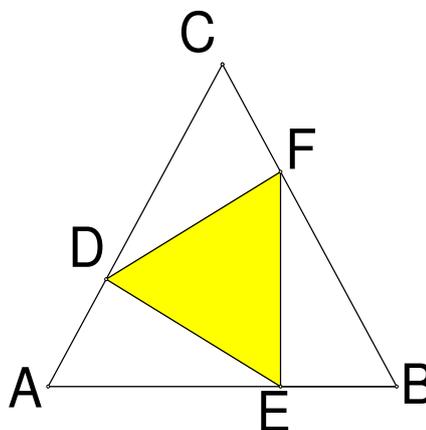
45. * Find the positive integer n such that

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{101 \cdot 102}\right].$$

(A) 98   (B) 99   (C) 100   (D) 101   (E) 102

46. Let x, y and z be three different real numbers such that

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Which is the value of $x^2 y^2 z^2$?

(A) 1   (B) 2   (C) 3   (D) 4   (E) 5

47. ∗ In a right triangle the legs a and b, measured in a certain unit, are positive integers such that $a > b$. Knowing that the hypothenuse, measured with the same unit, is $c = 2014$, what is $a - b$?

(A) 645   (B) 646   (C) 647   (D) 648   (E) 649

48. ∗ In the accompanying figure we have a square $ABCD$ and M is the mid-point of the side $\overline{BC}$. If points E and F are chosen at random with uniform distribution on the sides $\overline{AB}$ respectively $\overline{CD}$, what is the probability that the angle $\angle EMF$ is acute?

(A) $\dfrac{1 - \ln 2}{2}$ (B) $\dfrac{2 - \ln 3}{3}$ (C) $\dfrac{3 - \ln 4}{4}$

(D) $\dfrac{4 - \ln 5}{5}$ (E) $\dfrac{5 - \ln 6}{6}$
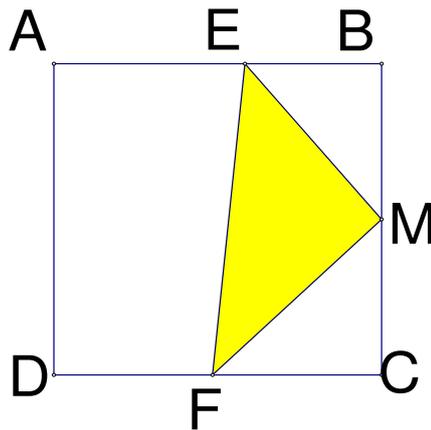
49. In triangle $\triangle ABC$ the angle $\angle ABC = 60°$. Let P be an interior point such that $\angle APB = \angle BPC = \angle CPA$, $PA = 8$ cm, and $PC = 6$ cm. What is the length of PB?

(A) $\sqrt{3}$ cm   (B) $2\sqrt{3}$ cm   (C) $3\sqrt{3}$ cm

(D) $4\sqrt{3}$ cm   (E) $5\sqrt{3}$ cm
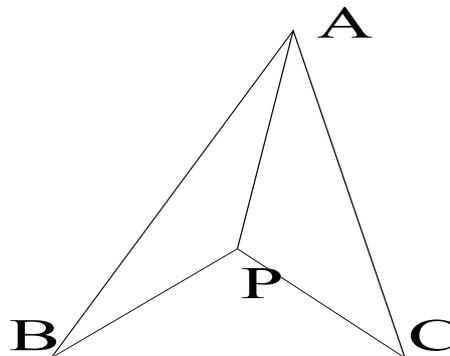
50. Let f be a positive continuous function defined on the real numbers which satisfies

$$f(x) = 3 + \int_0^x \frac{tf(t)}{1+t^2} dt,$$

for all real numbers x. What is $f(4/3)$?

(A) 1   (B) 2   (C) 3   (D) 4   (E) 5