Solutions of the Forty-fourth Annual Columbus State Invitational Mathematics Tournament

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1. How many positive integer divisors does 2018 have?

   (A) 2 (B) 4 (C) 6 (D) 8 (E) 10

   Answer: B

   We have $2018 = 2 \cdot 1009$, so the number of positive divisors is 4.

2. How many of the first 2018 positive integers are perfect squares?

   (A) 44 (B) 45 (C) 43 (D) 42 (E) 41

   Answer: A

   We have that $44^2 < 2018 < 45^2$, so there are 44 perfect squares smaller than 2018.

3. For integers a and b, we define $a \otimes b$ by the formula

$$a \otimes b = \text{gcf}(a, b) \cdot \text{lcm}(a, b),$$

   where "gcf" and "lcm" are the greatest common factor and least common multiple of a and b, respectively. Find the value of $18 \otimes 75$.

   (A) 675 (B) 270 (C) 1080 (D) 450 (E) 1350

Answer: E

In general, we have that $\text{gcf}(a,b)\cdot\text{lcm}(a,b) = a\cdot b$, so in our case $18 \otimes 75 = 18\cdot 75 = 1350$.

4. $*$ When a certain solid substance melts, its volume increases by $\frac{1}{7}$. By how much does its volume decrease when it solidifies again?

(A) $\frac{1}{7}$ (B) $\frac{1}{8}$ (C) $\frac{1}{9}$ (D) $\frac{1}{10}$ (E) $\frac{1}{11}$

Answer: B

If x is the initial volume, when the solid melts, its volume is $x + \frac{1}{7}x = \frac{8x}{7}$. When the substance solidifies, the substance's volume returns to x so it drops by $\frac{1}{8}$. Indeed, $\frac{8x}{7} - \frac{1}{8}\cdot\frac{8x}{7} = x$.

5. The equation

$$2017 + 2018i = (3 - 2i)(x + yi)$$

has a solution in positive integers x and y. What is $2y - 10x$?

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Answer: D

We have $2017 + 2018i = (3x + 2y) + i(3y - 2x)$. This means that $3x + 2y = 2017$ and $3y - 2x = 2018$, so $x = 155$ and $y = 776$. Therefore, $2y - 10x = 2$.

6. If 45 is the sum of n consecutive positive integers, what is the largest possible value on n?

(A) 3 (B) 5 (C) 7 (D) 9 (E) 11

Answer: D

The largest possible value is 9 since $1 + 2 + 3 + \cdots + 7 + 8 + 9 = 45$.

7. How many triples (x, y, z), of integer numbers, $x > y > z > 1$, satisfy the inequality

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: C

We have $z \geq 2$, $y \geq 3$ and $x \geq 4$. It is easy to see that there are no solutions if $z \geq 3$. If $z = 2$, we have $y = 3$, $x = 4$ and $y = 3$, $x = 5$. Thus, there only two solutions: $(2, 3, 4)$ and $(2, 3, 5)$.

8. How many 2-digit positive integers can be represented as the sum of different powers of 2?

(A) 30 (B) 45 (C) 60 (D) 75 (E) 90

Answer: E

Every positive integer can be represented as a sum of different powers of 2 (see base 2 representation) so, in particular, all 2-digit integers have such a representation.

9. $*$ For what value of the real number a does the system of equations

$$\begin{cases} x^2 + y^2 = z \\ x + y + z = a \end{cases}$$

have a unique solution in the set of real numbers?

(A) $\frac{1}{4}$ (B) $\frac{1}{5}$ (C) $\frac{1}{2}$ (D) $-\frac{1}{3}$ (E) $-\frac{1}{2}$

Answer: E

Combining the equations, we get $x^2+y^2+x+y = a$, so $\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = a + \frac{1}{2}$. This equation has a unique solution if and only if $a = -\frac{1}{2}$. This implies that the unique solution to the system is $x = y = -\frac{1}{2}$ and $z = \frac{1}{2}$, so the answer is E.

10. Find the sum of the real numbers x and y which satisfy the equation

$$\sqrt{4x^2 - 12x + 25} + \sqrt{4x^2 + 12xy + 9y^2 + 64} = 12.$$

(A) $\frac{7}{2}$ (B) $\frac{3}{2}$ (C) $\frac{9}{2}$ (D) $\frac{1}{2}$ (E) $\frac{5}{2}$

Answer: D

We have that

$$\sqrt{4x^2 - 12x + 25} + \sqrt{4x^2 + 12xy + 9y^2 + 64} =$$

$$\sqrt{(2x-3)^2 + 16} + \sqrt{(2x+3y)^2 + 64} \geq \sqrt{16} + \sqrt{64} = 12.$$

The equality occurs if and only if $2x - 3 = 0$ and $2x + 3y = 0$. This implies that $x = \frac{3}{2}$ and $y = -1$, thus $x + y = \frac{1}{2}$, so the answer is D

11. $*$ The positive real numbers x and y satisfy the equation

$$x^2 - 2xy - 3y^2 = 0.$$

What is the value of $\dfrac{x+2y}{x-y}$?

(A) 0 (B) 1 (C) $\dfrac{5}{2}$ (D) 3 (E) 4

Answer: C

The equation can be written as $(x-y)^2 - (2y)^2 = 0$, which factors as $(x-3y)(x+y) = 0$. Since x and y are positive, we have $x = 3y$, so $\frac{x+2y}{x-y} = \frac{5y}{2y} = \frac{5}{2}$.

12. Three distinct prime numbers p, q and r are chosen in such a way the number

$$p^4 + q^4 + r^4 - 3^5$$

is also a prime. What is the minimum possible value of $|pq - r|$?

(A) 2 (B) 1 (C) 4 (D) 0 (E) 3

Answer: B

Clearly $|pq - r| \geq 1$. We also have that $2^4 + 3^4 + 5^4 - 3^5 = 479$, which is prime. Since $|2 \cdot 3 - 5| = 1$, we get that the smallest possible value is 1.

13. $*$ What is the units digit of 2018^{2018}?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Answer: C

It is easy to see that for each positive integer k, 8^{4k} has the units digit 6. This implies that 8^{2018} has the units digit 4, so 2018^{2018} has the units digit 4.

14. $*$ There is only one 4-digit integer n for which $\sqrt{3\sqrt{2\sqrt{n}}}$ is an integer. Find the sum of the digits of n.

(A) 17 (B) 18 (C) 19 (D) 20 (E) 21

Answer: B

Note that $\sqrt{3\sqrt{2\sqrt{n}}} = \sqrt[8]{3^4 \cdot 2^2 \cdot n}$, so n is a multiple of $3^4 \cdot 2^6 = 5184$. Since there are no other multiples which have 4 digits, we have that $n = 5184$. The sum of its digits is 18.

15. $*$ Let x, y and z be positive integers such that

$$x + \frac{1}{y + \frac{1}{z}} = \frac{26}{21}.$$

What is the value of xyz?

(A) 20 (B) 24 (C) 28 (D) 32 (E) 36

Answer: A

Clearly $x = 1$, so we have to solve the equation $\dfrac{z}{yz+1} = \dfrac{5}{21}$. This implies that $y = 4$ and $z = 5$, so the product $xyz = 20$.

16. $*$ Twelve marbles are placed in three boxes such that each box contains a red marble, a blue marble, a black marble, and a white marble. If we pick at random one marble from each box, what is the probability that exactly two marbles are red?

(A) $\dfrac{9}{64}$ (B) $\dfrac{1}{8}$ (C) $\dfrac{3}{32}$ (D) $\dfrac{5}{64}$ (E) $\dfrac{1}{16}$

Answer: A

There are 64 possible outcomes of which 9 have exactly two red marbles, so the answer is $\dfrac{9}{64}$.

17. How many permutations (x_1, x_2, x_3, x_4) of the set of integers $\{1, 2, 3, 4\}$ have the property that the sum $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$ is not divisible by 3?

(A) 6 (B) 8 (C) 12 (D) 14 (E) 16

Answer: B

Since $x_1x_2+x_2x_3+x_3x_4+x_4x_1 = (x_1+x_3)(x_2+x_4)$, we have to count those permutations such that x_1+x_3 and x_2+x_4 are not multiples of 3. There are 8 such possibilities.

18. The numbers a and b are chosen from the set $\{1, 2, \ldots, 26\}$, such that the product ab is equal to the sum of the remaining numbers. What is the value of $|a-b|$?

(A) 3 (B) 6 (C) 8 (D) 10 (E) 13

Answer: B

Since $1+2+\cdots+26 = 351$, we have that $ab = 351-a-b$. This implies that $(a+1)(b+1) = 352 = 2^5 \cdot 11$, so either $a+1$ or $b+1$ is a multiple of 11. Without losing the generality, we may assume that $a+1$ is a multiple of 11. Since a is smaller than 26, we have that $a = 10$, or $a = 21$. But if $a = 10$, we get $b = 31$, which is impossible since $b \le 26$. Therefore, $a = 21$ and $b = 15$, so $|a-b| = 6$.

19. How many triangles ΔABC with $\angle ABC = 90°$ and $AB = 10$ exist such that all sides have integer lengths?

(A) infinitely many (B) 3 (C) 2

(D) 1 (E) none

Answer: D

If x is the length of hypothenuse AC and y is the length of side BC, we have $x^2 = y^2+100$. This implies that $(x-y)(x+y) = 100$. Since both x and y have to be integers, we have that $x-y$ and $x+y$ are factors of 100. Also, we have that $x > y$ and $x-y$ and $x+y$ have the same parity. The only possibilities are $x-y = 2$ and $x+y = 50$, which lead to the unique solution $x = 26$ and $y = 24$. Therefore the answer is D.

20. Find the product of the solutions of the equation

$$\sqrt[3]{(x+1)^2} + \sqrt[3]{(x-1)^2} = \frac{5}{2}\sqrt[3]{x^2-1}.$$

(A) $-\frac{36}{121}$ (B) $-\frac{16}{81}$ (C) $-\frac{9}{25}$ (D) $-\frac{25}{36}$ (E) $-\frac{81}{49}$

Answer: E

Note that $x = 1$ and $x = -1$ are not solutions. Dividing the equation by $\sqrt[3]{x^2-1}$ and simplifying, results in the equivalent equation

$$\sqrt[3]{\frac{x+1}{x-1}} + \sqrt[3]{\frac{x-1}{x+1}} = \frac{5}{2}.$$

Setting $t = \sqrt[3]{\frac{x+1}{x-1}}$, we have to solve $t + \frac{1}{t} = \frac{5}{2}$. This reduces to the quadratic equation $t^2 - \frac{5}{2}t + 1 = 0$, which implies that $t = 2$ or $t = \frac{1}{2}$. In the first case, we get $x = \frac{9}{7}$ and in the second case we get $x = -\frac{9}{7}$.

21. $*$ Find the number of integers n such that $\dfrac{n^2}{n+2018}$ is an integer.

(A) 12 (B) 14 (C) 16 (D) 18 (E) 20

Answer: D

Note that $\frac{n^2}{n+2018} = n - 2018 + \frac{2018^2}{n+2018}$, so the problem is equivalent to finding all integers n such that $n + 2018$ divides 2018^2. The prime factorization of 2018^2 is $2018^2 = 2^2 \cdot 1009^2$, so 2018^2 has exactly 18 integer factors. Thus, we have the following 18 equations: $n + 2018 = \pm1, \pm2, \pm4, \pm1009, \pm1009^2, \pm2 \cdot 1009, \pm2^2 \cdot 1009, \pm2 \cdot 1009^2, \pm2^2 \cdot 1009^2$. Thus, we get 18 solutions for n, such that $\frac{n}{n+2018}$ is an integer so the answer is D.

22. What is the remainder when 2017^{2018} is divided by 13?

(A) 1 (B) 5 (C) 3 (D) 2 (E) 4

Answer: E

One checks that 2017 has remainder 2 when divided by 13. Also, the smallest power of 2 with the property that 2^m has remainder 1 when divided by 13 is 12. This means that the remainder of the division 2017^{2018} by 13 is the same as that of 2^{2018} by 13. Also, $2^{2018} = (2^{12})^{168} \cdot 2^2 = (4096)^{168} \cdot 4 = (13 \cdot 315 + 1)^{168} \cdot 4$, so the reminder is 4.

23. Three regular dice are rolled. The probability that the numbers thrown have the least common multiple equal to 60 is equal to $\dfrac{m}{n}$, for some relatively prime positive integers m and n. What is the value of $\dfrac{n-2m}{m+3}$?

(A) 5 (B) 2 (C) 4 (D) 1 (E) 3

Answer: C

To get the least common multiple equal to 60, one must roll $(5,6,4),(5,3,4)$, or any permutation of these numbers. Therefore there are 12 possible triplets which give a common multiple of 60. Since there are $6^3 = 216$ possible outcomes, the probability to have the least common multiple equal to 60 is $\frac{12}{216} = \frac{1}{18}$. This means that $\frac{n-2m}{m+3} = 4$, so the answer is C.

24. Vlad is playing on the mall escalators. One escalator goes up, one goes down, and one is out of service; otherwise, they're all identical. The up and down escalators go at the same speed. You can assume that Vlad always runs at the same speed. Vlad can run up the up escalator in 6 seconds. He can run up the down escalator in 30 seconds. How long does it take him to run up the out-of-service escalator?

(A) 10 s (B) 12 s (C) 14 s (D) 16 s (E) 18 s

Answer: A

Let s_v be Vlad's speed, s_e the speed of the up and down escalators, and d the length of each escalator. Then we have $d = 6(s_v + s_e)$ and $d = 30(s_v - s_e)$. Multiply the first equation by 5 and add it to the second. We get $6d = 60s_v$, so $d = 10s_v$, thus it takes Vlad 10 seconds to run up the out-of-service escalator.

25. How many triples (x, y, z), of integer numbers, $x > y > z > 1$, satisfy the inequality

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1?$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: C

We have $z \geq 2$, $y \geq 3$ and $x \geq 4$. It is easy to see that there are no solutions if $z \geq 3$. If $z = 2$, we have $y = 3$, $x = 4$ and $y = 3$, $x = 5$. Thus, there only two solutions: $(2,3,4)$ and $(2,3,5)$.

26. $*$ The function f satisfies the relations

$$f(1) = 1 \text{ and } f(m+n) = f(m) + f(n) + mn,$$

for all positive integers m and n. What is the value of $f(200)$?

(A) 20000 (B) 20010 (C) 20100 (D) 21000 (E) 21100

Answer: C

For $n = 1$, we get $f(m+1) = f(m) + m + 1$ for all positive integers m. This implies that $f(m+1) = f(m-1)+m+(m+1) = \cdots = 1+2+\cdots+m+(m+1) = \frac{(m+1)(m+2)}{2}$. Now, we get that $f(200) = \dfrac{200 \cdot 201}{2} = 20100$.

27. $*$ If x, y, and z are real numbers such that $x, y, z > 1$, what is the smallest possible value of

$$\log_{xy} z + \log_{yz} x + \log_{xz} y?$$

(A) $\dfrac{7}{5}$ (B) $\dfrac{3}{2}$ (C) $\dfrac{8}{5}$ (D) $\dfrac{29}{20}$ (E) $\dfrac{4}{3}$

Answer: B

Note that for any positive real number a, we have $a + \frac{1}{a} \geq 2$. Also, recall that if a, b are positive an not equal to 1, we have $\log_a b = \frac{1}{\log_b a}$. Taking these into account we have the inequalities

$$\begin{aligned}\log_{xy}(yz) + \log_{yz}(xy) \geq 2\\ \log_{yz}(zx) + \log_{zx}(yz) \geq 2\\ \log_{yx}(zx) + \log_{zx}(yx) \geq 2.\end{aligned}$$

Adding these inequalities and using the properties of logs, we get that $\log_{xy} z + \log_{yz} x + \log_{xz} y \geq 3/2$. Note that if $x = y = z$, we get the equality so the answer is B.

28. Find the sum of all real numbers m with the property that the equation

$$x^2 - x + m = 0$$

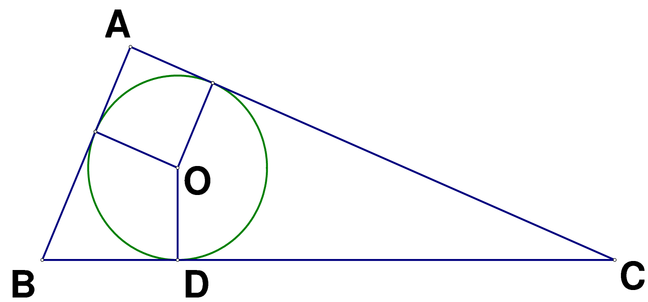
has two solutions, x_1 and x_2, which satisfy the equation $x_1^5 + x_2^5 = 211$.

(A) 2 (B) 4 (C) 3 (D) 1 (E) 5

Answer: D

We have that $x_1+x_2 = 1$ and $x_1x_2 = m$. This implies that $x_1^2+x_2^2 = (x_1+x_2)^2-2x_1x_2 = 1 - 2m$ and $x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2) = 1 - 3m$. Now, $x_1^5 + x_2^5 = (x_1^2 + x_2^2)(x_1^3 + x_2^3) - x_1^2x_2^2(x_1 + x_2) = (1 - 2m)(1 - 3m) - m^2$, so we get the equation $5m^2 - 5m - 210 = 0$, which factors as $5(m - 7)(m + 6) = 0$. This implies that $m = 7$ or $m = -6$, so the sum of the values of m is 1.

29. In the accompanying figure, ΔABC is a right triangle with hypothenuse $BC = 2018$ cm and $AC = 1680$ cm. The circle inscribed in this triangle has radius $r = OD$. What is the value of r, in centimeters?



(A) 390 (B) 380 (C) 370

(D) 360 (E) 350

Answer: A

By the Pythagorean theorem we have, $AB^2 = 2018^2 - 1680^2 = 1118^2$. On the other hand, the radius is $r = (b + c - a)/2 = 390$.

30. $*$ The function f satisfies the conditions $f(4) = 6$ and $xf(x) = (x - 3)f(x + 1)$, for all integers x. What is the value of the product

$$f(2018) \cdot f(2015) \cdot f(2012) \cdots f(8) \cdot f(5)?$$

(A) 0 (B) 2017 (C) 2017! (D) 2018 (E) 2018!

Answer: C

We have that $f(x+1) = \dfrac{x}{x-3} f(x) = \dfrac{x}{x-3} \cdot \dfrac{x-1}{x-4} \cdot f(x-1) = \cdots = \dfrac{x}{x-3} \cdot \dfrac{x-1}{x-4} \cdots \dfrac{4}{1} \cdot f(4)$. This implies that $f(x+1) = x(x-1)(x-2)$, so $f(2018) \cdot f(2015) \cdots f(5) = 2017!$.

31. Let x and y be positive integers such that $\dfrac{2018!}{7^x \cdot 13^y}$ is an integer. What is the largest possible value of $x + y$?

(A) 480 (B) 485 (C) 490 (D) 495 (E) 500

Answer: E

The exponent of 7 in 2018! is 334. Indeed, we have

$$\lfloor \tfrac{2018}{7} \rfloor + \lfloor \tfrac{2018}{49} \rfloor + \lfloor \tfrac{2018}{343} \rfloor + \lfloor \tfrac{2018}{2401} \rfloor = 288 + 41 + 5 + 0 = 334.$$

Similarly, the exponent of 13 in 2018! is 166 because

$$\lfloor \tfrac{2018}{13} \rfloor + \lfloor \tfrac{2018}{169} \rfloor + \lfloor \tfrac{2018}{2197} \rfloor = 155 + 11 + 0 = 166.$$

The maximum value of $x+y$ is 500.

32. Find the sum of the digits of the unique solution of the equation

$$x^{\log_{100}(x-1)} + 100(x-1)^{\log_{100} x} = 101x^2.$$

(A) 2 (B) 6 (C) 10 (D) 14 (E) 18

Answer: A

The conditions for the existence of the logarithms and the exponential functions imply that $x > 1$. Note that $x^{\log_{100}(x-1)} = (x-1)^{\log_{100} x}$, so the equation becomes $101x^{\log_{100}(x-1)} = 101x^2$. This implies that $x^{\log_{100}(x-1)} = x^2$, so $\log_{100}(x-1) = 2$ or $x = 10001$. Therefore, the sum of the digits of x is 2.

33. $*$ If x is a positive real number, what is the minimum possible value of the expression

$$E(x) = \frac{\left(x+\frac{1}{x}\right)^4 - \left(x^4 + \frac{1}{x^4}\right)}{\left(x+\frac{1}{x}\right)^3 - \left(x^3 + \frac{1}{x^3}\right)}?$$

(A) $\dfrac{2}{3}$ (B) $\dfrac{4}{3}$ (C) $\dfrac{5}{3}$ (D) $\dfrac{7}{3}$ (E) $\dfrac{8}{3}$

Answer: D

Let $y = x + \frac{1}{x}$ and note that $y \geq 2$. Successively rasing the identity to the second, third, and fourth powers, gives us $x^2 + \frac{1}{x^2} = y^2 - 2$, and $x^3 + \frac{1}{x^3} = y^3 - 3y$, and $x^4 + \frac{1}{x^4} = y^4 - 4y^2 + 2$. This implies that $E(x) = \frac{4y^2-2}{3y} = \frac{4}{3}y - \frac{2}{3y}$. It is easy to see that the function $g(y) = \frac{4}{3}y - \frac{2}{3y}$ is increasing for $y \geq 2$, so the minimum value is attained at $y = 2$. This means that $x = 1$, so the minimum of $E(x)$ is $\frac{7}{3}$.

34. $*$ Let x, y, z be real numbers such that

$$3^x + 3^y + 3^z + 3^{-x} + 3^{-y} + 3^{-z} = 10.$$

If $E(x,y,z) = 3^x + 3^y + 3^z$, what is the difference between the largest and the smallest possible values of $E(x,y,z)$?

(A) 8 (B) 7 (C) 9 (D) 6 (E) 10

Answer: A

If $F(x, y, z) = 3^{-x} + 3^{-y} + 3^{-z}$, we have that $E + F = 10$ and that $E \cdot F \geq 9$. This implies that $E(10 - E) \geq 9$, so $1 \leq E \leq 9$. Note that $x = y = z = 1$ is a solution of the equation and that $E(1, 1, 1) = 9$. Also, note that $x = y = z = -1$ is also a solution of the equation and that $E(-1, -1, -1) = 1$, so the difference between the largest and smallest possible values of E is 8.

35. $*$ Let x be a real number such that

$$\sec x - \tan x = \frac{1}{2}.$$

Find the value of $\cos x$.

(A) $\frac{3}{4}$ (B) $\frac{2}{3}$ (C) $\frac{4}{5}$ (D) $\frac{6}{7}$ (E) $\frac{1}{2}$

Answer: C

We have $\sec x = \frac{1}{2} + \tan x$. By squaring both sides we obtain $\sec^2 x = \frac{1}{4} + \tan x + \tan^2 x$. Since $\sec^2 x = 1 + \tan^2 x$, we get that $\tan x = \frac{3}{4}$. If follows that $\sec x = \frac{5}{4}$, so $\cos x = \frac{4}{5}$.

36. $*$ A line is parallel to the line $y = \dfrac{5}{4}x + \dfrac{95}{4}$, intersects the x-axis and y-axis at points A and B, respectively, and passes through the point $(-1, -25)$. How many points with integer coordinates are there on the line segment AB (including points A and B)?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: E

The slope of the line is $m = \frac{5}{4}$, so $y = \frac{5}{4}x + b$. Since $(-1, -25)$ is on the line, we get $b = -\frac{95}{4}$. We get that the y-intercept is $B(0, \frac{95}{4})$ and the x-intercept is $A(19, 0)$. Now, we have to find all integers x, such that $0 \leq x \leq 19$ and $\frac{5}{4}x - \frac{95}{4}$ is an integer. This is equivalent to $5x - 95$ is a multiple of 4. Since $5x - 95 = 4(x - 24) + (x + 1)$, we need to determine the values or x such that $x + 1$ is a multiple of 4. These values are $x = 3, 7, 11, 15, 19$, so there are 5 points with integer coordinates on the line segment AB.

37. $*$ A sphere is inscribed in a cube, and a smaller cube is inscribed within the sphere. What is the ratio of the volume of the large cube to the volume of the small cube?

(A) $4\sqrt{2}$ (B) $6\sqrt{3}$ (C) $2\sqrt{2}$ (D) $2\sqrt{3}$ (E) $3\sqrt{3}$

Answer: E

If the large cube has side a, the sphere has diameter a, so the diagonal of the smaller cube is also equal to a. Since the diagonal in a cube of side x is $x\sqrt{3}$, we get that the side of the small cube is $\frac{a}{\sqrt{3}}$. This means that the ratio of the two volumes is $3\sqrt{3}$.

38. Suppose that real numbers x and y satisfy the equation

$$4x^2 - 6xy + 4y^2 = 7.$$

Let $S = x^2 + y^2$ and S_{min} and S_{max} denote the minimum and maximum values of S, respectively. Find the value of $S_{min} + S_{max}$.

(A) 3 (B) 4 (C) 6 (D) 8 (E) 9

Answer: D

If $x = r \sin\alpha$ and $y = r\cos\alpha$, the equation becomes

$$4r^2 - 6r^2 \sin\alpha\cos\alpha = 7.$$

Therefore, $S = r^2 = \frac{7}{4-3\sin 2\alpha}$ and $S_{min} + S_{max} = 1 + 7 = 8$.

39. $*$ Two of the altitudes of a triangle are 10 cm and 11 cm. Which of the following **can not** be the length of the third altitude?

(A) 6 cm (B) 5 cm (C) 8 cm (D) 7 cm (E) 9 cm

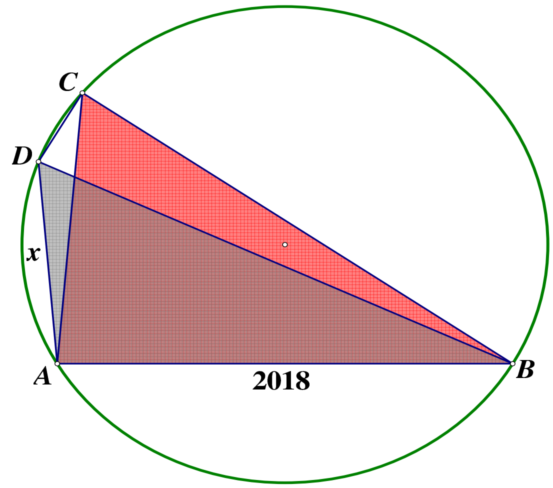
Answer: B

Let a and b be the sides corresponding to the altitudes of length 10 cm and 11 cm respectively. Let c be the length of the third side and h_c the length of the third altitude. The area of the triangle is $A = \frac{10 \cdot a}{2} = \frac{11 \cdot b}{2} = \frac{h_c \cdot c}{2}$, so we have $a = \frac{2A}{10}$, $b = \frac{2A}{11}$, and $c = \frac{2A}{h_c}$. The sum of any two of the sides of the triangle has to be greater than the third side, so we have that $h_c = 5$ is impossible ($\frac{1}{10} + \frac{1}{11} < \frac{1}{5}$).

40. In the accompanying figure, ABCD is a cyclic quadrilateral (inscribed in a circle) in which $AB = 2018$ cm, $BD = BC = 2322$ cm, and $AC = 1330$ cm. The length of $x = AD$ (in centimeters) is a three-digit number (in base 10). What is the units digit of x?

(A) 5 (B) 4 (C) 3

(D) 2 (E) 1



Answer: D

From the Law of Cosines, in triangle ΔABC, we have that

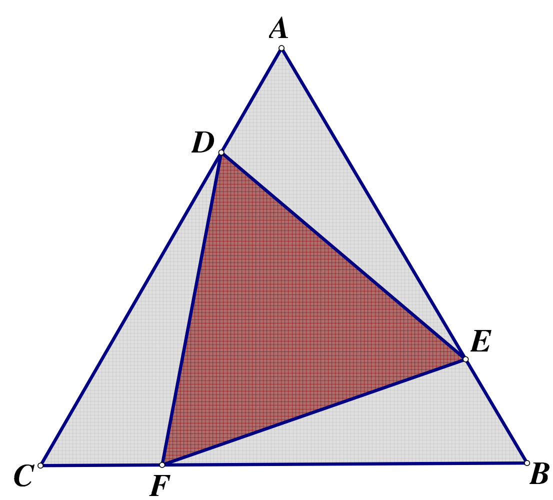
$$\cos(\angle BCA)) = \frac{1330^2 + 2322^2 - 2018^2}{2 \cdot 1330 \cdot 2322} = \frac{1}{2}.$$

This implies that $\angle BCA = 60°$. Since $ABCD$ is a cyclic quadrilateral, we get that $\angle ADB = 60°$. If we use the Law of Cosines in triangle ΔADB, we have $AB^2 = DB^2 + DA^2 - 2 \cdot DB \cdot DA \cdot \cos(60°)$, so x is a solution of the quadratic equation $x^2 - 2322x + 1319360 = 0$. This implies $x = 992$ or $x = 1330$. Since x is a three-digit number, we have $x = 992$, therefore the units digit of x is 2.

41. In the equilateral triangle ΔABC, the equilateral triangle ΔDEF is inscribed in such a way $\dfrac{CD}{DA} = \dfrac{AE}{EB} = \dfrac{BF}{CF} = 3$. The ratio between the areas of the triangles ΔABC and ΔDEF is $\dfrac{m}{n}$, for some relatively prime positive integers m and n. What is $5m - 11n$?

(A) 1 (B) 2 (C) 3

(D) 4 (E) 5



Answer: C

It is easy to see that $AD = BE = CF = x$ and $CD = AE = BF = 3x$, for some positive real number x. The area of triangle ΔABC is $\frac{16x^2\sqrt{3}}{4}$. The area of triangles ΔADE, ΔBEF, and ΔCDF is equal to $\frac{3x^2\sqrt{3}}{4}$. This implies that the area of $\Delta DEF = \frac{7x^2\sqrt{3}}{4}$. Therefore the ratio between the areas of the triangles ΔABC and ΔDEF is $\frac{16}{7}$, so $5m - 11n = 80 - 77 = 3$.

42. How many integer triples (x, y, z) satisfy the equation

$$x^2 + y^2 + z^2 = 2^{2018}(x + y + z)?$$

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Answer: D

Since $x^2 + y^2 + z^2$ is even, x, y, z are all even or two of them are odd. If $x = 2x_1 + 1$, $y = 2y_1 + 1$, and $z = 2z_1$, we get that $4(x_1^2 + x_1 + y_1^2 + y_1 + z_1^2) + 2 = 2^{2019}(x_1 + y_1 + z_1)$, which is impossible. Thus, $x = 2x_1$, $y = 2y_1$, $z = 2z_1$, and we have the equation $x_1^2 + y_1^2 + z_1^2 = 2^{2017}(x_1 + y_1 + z_1)$. Reiterating the above argument, we get that the solutions of the equation are $x, y, z \in \{0, 2^{2018}\}$. Therefore, there are 8 different triples (x, y, z), which satisfy the equation.

43. For a positive integer n, written in base 10, we denote by $p(n)$ the product of its digits. What is the sum of the digits of n, if it satisfies the equation

$$10p(n) = n^2 + 6n - 2095?$$

(A) 15 (B) 12 (C) 6 (D) 18 (E) 9

Answer: E

Note that $n^2 + 6n - 2095 \geq 0$, so $n \geq 43$. Also, note that $p(n) \leq n$, so we have the inequality $n^2 - 4n - 2095 \leq 0$, so $0 \leq n \leq 47$. Finally, since $n^2 + 6n$ is a multiple of 5, the only possibilities are $n = 44$ and $n = 45$. Only $n = 45$ satisfies the given equation, so the answer is 9.

44. Two positive numbers a and b are chosen in such a way that $\dfrac{b}{a} = e \approx 2.71828$ (the Euler number). Two points of coordinates x and y are chosen at random from the interval $[0, b]$. The probability that the geometric average of x and y is greater than a is equal to $1 - \dfrac{m}{e^2}$. What is the value of m ?

(A) 3 (B) 1 (C) 4 (D) 2 (E) 5

Answer: A

We need to have $xy \geq a^2$. The probability that $y \geq \frac{a^2}{x}$ is equal to ratio between the area of the region above the curve $y \geq \frac{a^2}{x}$, but which lies inside the square $[0,b] \times [0,b]$, and the area of the square $[0,b] \times [0,b]$. The first area is equal to

$$\int_{\frac{a^2}{b}}^{b} \left(b - \frac{a^2}{x}\right) dx = b^2 - 3a^2.$$

On the other hand, the area of the square $[0,b] \times [0,b]$ is b^2, so the ratio is equal to $\frac{b^2-3a^2}{b^2} = 1 - \frac{3}{e^2}$.

45. The sequence $\{x_n\}_{n\geq 1}$ is defined by the formula

$$x_n = \sin(\pi\sqrt{4n^2+2n+1}),$$

for all positive integers n. Find the value of $\lim\limits_{n\to\infty} x_n$.

(A) $\dfrac{1}{2}$ (B) 1 (C) $\dfrac{\sqrt{2}}{2}$ (D) 0 (E) $\dfrac{\sqrt{3}}{2}$

Answer: B

Using that sine is periodic of period 2π, we have that

$$\sin(\pi\sqrt{4n^2+2n+1}) = \sin[\pi \cdot (\sqrt{4n^2+2n+1} - 2n)] = \sin\left[\frac{(2n+1)\pi}{\sqrt{4n^2+2n+1}+2n}\right] =$$

$$\sin\left[\frac{2n\pi\left(1+\frac{1}{2n}\right)}{2n\left(\sqrt{1+\frac{1}{2n}+\frac{1}{4n^2}}+1\right)}\right] = \sin\left[\frac{\pi\left(1+\frac{1}{2n}\right)}{\left(\sqrt{1+\frac{1}{2n}+\frac{1}{4n^2}}+1\right)}\right].$$

Therefore,

$$\lim_{n\to\infty} \sin(\pi\sqrt{4n^2+2n+1}) = \lim_{n\to\infty} \sin\left[\frac{\pi\left(1+\frac{1}{2n}\right)}{\left(\sqrt{1+\frac{1}{2n}+\frac{1}{4n^2}}+1\right)}\right] = \sin\frac{\pi}{2} = 1.$$

46. Consider the sequence $\{x_n\}_{n\geq 1}$ such that

$$x_1 = 1 \text{ and } x_{n+1} = x_n + \frac{1}{2x_n}, \text{ for } n \geq 1.$$

Find $\lim\limits_{n\to\infty} \dfrac{x_n}{\sqrt{n}}$.

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Answer: D

Note that x_n is increasing, since $x_{n+1} > x_n > 1$, for all n. Also, $\lim_{n\to\infty} x_n = \infty$. Otherwise, if we assume x_n convergent to L, we would have $L = L + \frac{1}{2L}$, which is impossible. Now, use Cesaro-Stolz to obtain

$$\lim_{n\to\infty} \frac{x_n^2}{n} = \lim_{n\to\infty} \frac{x_{n+1}^2 - x_n^2}{(n+1)-n} = \lim_{n\to\infty} \left(x_n + \frac{1}{2x_n}\right)^2 - x_n^2 = \lim_{n\to\infty} \left(1 + \frac{1}{4x_n^2}\right) = 1.$$

Therefore, $\lim_{n\to\infty} \frac{x_n}{\sqrt{n}} = 1$.

47. Let a and b be positive real numbers. If the equation

$$x + \ln(ab) = \ln(x+a) + \ln(x+b)$$

has only one real solution x, what is the value of $\frac{1}{a} + \frac{1}{b}$?

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Answer: E

The function $f(x) = x + \ln(ab) - (\ln(x+a) + \ln(x+b))$ has a derivative equal to $f'(x) = 1 - \frac{1}{x+a} - \frac{1}{x+b}$. Since $f(0) = 0$, then $f'(0) = 0$. One can show that we have more than one solution if $f'(0) \neq 0$.

48. Two points are chosen at random on a circle (uniform distribution on the circumference) of radius 1. The probability that the distance between them is more than 1 is equal to $\frac{m}{n}$, for some relatively prime positive integers m and n. What is the value of $n - m$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: A

If A and B are points on the circumference of the circle of center O and radius 1, the distance between A and B is more than 1 if and only if the central angle $\angle AOB$ is bigger that $60°$. This means that for a given point A, the point B would have to be outside of a $120°$ central angle bisected by OA. Therefore, the probability is $\frac{2}{3}$ and the answer is A.

49. Two positive numbers a and b are chosen in such a way that $\frac{b}{a} = e \approx 2.71828$ (the Euler number). Two points of coordinates x and y are chosen at random from the

interval $[0, b]$. The probability that the geometric average of x and y is greater than a is equal to $1 - \frac{m}{e^2}$. What is the value of m ?

(A) 3 (B) 1 (C) 4 (D) 2 (E) 5

Answer: A

We need to have $xy \geq a^2$. The probability that $y \geq \frac{a^2}{x}$ is equal to ratio between the area of the region above the curve $y \geq \frac{a^2}{x}$, but which lies inside the square $[0, b] \times [0, b]$, and the area of the square $[0, b] \times [0, b]$. The first area is equal to

$$\int_{\frac{a^2}{b}}^{b} \left(b - \frac{a^2}{x}\right) dx = b^2 - 3a^2.$$

On the other hand, the area of the square $[0, b] \times [0, b]$ is b^2, so the ratio is equal to $\frac{b^2 - 3a^2}{b^2} = 1 - \frac{3}{e^2}$.

50. How many real numbers x, $0 \leq x \leq 2018$, are solutions of the equation

$$2^{\sin^4 x - \cos^2 x} - 2^{\cos^4 x - \sin^2 x} = \cos 2x?$$

(A) 1286 (B) 1285 (C) 1284 (D) 1283 (E) 1282

Answer: B

The equation is equivalent to

$$2^{\sin^4 x + \sin^2 x} - 2^{\cos^4 x + \cos^2 x} = 2\cos 2x = \cos^4 x + \cos^2 x - \sin^4 x - \sin^2 x.$$

This can be viewed as $2^a + a = 2^b + b$, for $a = \sin^4 x + \sin^2 x$ and $b = \cos^4 x + \cos^2 x$. The function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = 2^t + t$ is increasing, so it is also one-to-one. This means that $a = b$, so $\sin^4 x + \sin^2 x = \cos^4 x + \cos^2 x$, which is equivalent with $2\cos 2x = 0$. Therefore the solutions of the equation are of the form $x = \frac{(2k+1)\pi}{4}$, where k is an integer. Since $0 \leq x \leq 2018$, we get that $0 \leq k \leq 1284$, so there are 1285 solutions in the given interval.