TWENTY-SIXTH ANNUAL COLUMBUS INVITATIONAL MATHEMATICS TOURNAMENT

Sponsored by

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DEPARTMENT OF MATHEMATICS

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The Columbus State University Mathematics faculty welcome you to this year's tournament and to our campus. We hope that you will be successful on this test.

INSTRUCTIONS: This is a 90-minute, 50-problem, multiple choice examination. There are five (5) possible responses to each question. You are to select the one (1) "**best**" answer to each. In some instances this may be the closest approximation rather than an exact answer. You may mark on the test booklet and on the paper provided. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the choice you have made on the test booklet. After you have worked all of the problems that you can work, carefully transfer your answers to the score sheet. Darken completely the blank corresponding to the letter of your response to each question. Mark your answers boldly with a No. 2 pencil. If you must change an answer, completely erase your first choice and then record the new answer. Incomplete erasures and multiple marks for any question will be scored as an incorrect response. The examination will be scored on the basis of +12 for each correct answer, -3 for each incorrect selection and 0 for each omitted item. Each student will be given an initial score of +200.

Pre-selected problems will be used as tie-breakers for individual awards. These problems, designated with an asterisk (*), in order of consideration are: 50, 45, 49, 48, and 43.

Throughout this exam, AB will denote the line segment from point A to point B. $\overline{AB}$ will denote the length of AB. All pre-drawn geometric figures are not necessarily drawn to scale.

Review and check your score sheet carefully. Your student identification number and your school number must be encoded correctly on your score sheet.

When you complete your test, bring your pencil, scratch paper and answer sheet to the Test Monitor. You may leave the room after you have handed in your answer sheet. Please leave quietly so as not to disturb the other contestants. Please do not congregate outside the doors to the testing area. You may keep your copy of the test. Your sponsor will have a copy of the answers.

DO NOT OPEN TEST UNTIL INSTRUCTED TO DO SO

1. What is the area (in square inches) of a square if its circumscribed circle has diameter 20 inches?

   (a) 200   (b) 100   (c) 400   (d) 360   (e) 300

2. How many integers between 100 and 1000 are multiples of 7?

   (a) 130   (b) 136   (c) 132   (d) 134   (e) 128

3. Six times a certain positive number, added to its square, yields its cube. What is this number?

   (a) 0   (b) 3   (c) -2   (d) -3   (e) 2

4. How many three digit numbers less than 600 can be formed using the digits 8, 6, 4, and 2 (Digits may be used more than once)?

   (a) 64   (b) 128   (c) 32   (d) 16   (e) 48

5. Evaluate the expression $\dfrac{1}{81^{-\frac{1}{2}}}$.

   (a) $\dfrac{1}{9}$ (b) $-\dfrac{1}{9}$ (c) 9

   (d) -9 (e) none of these

6. Find the domain of the function $g(x) = \sqrt{3 - x}$.

   (a) $x \geq 3$ (b) $x \leq 0$ (c) $-3 < x < 3$

   (d) $x \leq 3$ (e) $x \geq 0$

7. Find the real value(s) of x for which $\sqrt{2 - 5x} = 5x$.

(a) $\dfrac{2}{5}$

(b) $-\dfrac{2}{5}$

(c) $\dfrac{1}{5}, \ -\dfrac{2}{5}$

(d) $\dfrac{1}{5}$

(e) none of these

8. Find the sum of all solutions of the equation $|2 - 4x| = 12$.

(a) -6

(b) 0

(c) 1

(d) $-\dfrac{5}{2}$

(e) 6

9. Determine the value of x such that $\dfrac{3x + 2}{5} - \dfrac{6x + 4}{3} = \dfrac{14}{3}$.

(a) $-\dfrac{44}{21}$

(b) 5

(c) $\dfrac{37}{12}$

(d) -4

(e) no solution

10. A right circular cylinder has a diameter of 6 inches and a height of 4 inches. Find the volume of the cylinder in cubic inches.

(a) 16π

(b) 24π

(c) 24

(d) 36

(e) 36π

11. The cost of a pair of pants was \$35.70. This included a 6% sales tax. How much were the pants excluding the sales tax?

(a) \$33.68
(b) \$30.00
(c) \$37.84
(d) \$33.56
(e) \$2.14

2

12. Find the digit, d, so that $49d84$ is divisible by 24.

    (a) 8   (b) 2   (c) 4

    (d) 6   (e) none of these

13. There are 26 students in Ms. Pietz's English class in Tokyo. All of the students speak English or Japanese and some speak both languages. If 18 of the students can speak English and 14 can speak Japanese, how many speak English only?

    (a) 6   (b) 8   (c) 14   (d) 18   (e) 12

14. A goat is tethered to an outside corner of a rectangular building that is 20 feet long and 6 feet wide with a rope that is 10 feet long. Calculate the grazing area in square feet.

    (a) 67π (b) 79π (c) 75π (d) 84π (e) 90π

15. If $f(x) = x^2 + bx + c$, and $f(2+t) = f(2-t)$ for all real numbers t, then which of the following is true?

    (a) $f(1) < f(2) < f(4)$ (b) $f(2) < f(1) < f(4)$ (c) $f(2) < f(4) < f(1)$

    (d) $f(4) < f(2) < f(1)$ (e) none of these

16. Determine which of the following is a solution of the equation $9^{-x} - 2 \cdot 3^{1-x} = 27$.

    (a) $x = 6$ (b) $x = 2$ (c) $x = 3$ (d) $x = -3$ (e) $x = -2$

17. Determine the number of different real pairs (x, y) that satisfy the system of equations
$$\begin{cases} x^2 + y & = 0 \\ x + y^2 & = 0 \end{cases}.$$

   (a) 0   (b) 2   (c) 1   (d) 3   (e) 4

18. Suppose that for some numbers a, b, and k we have the factorization
$$x^3 + kx^2 - x - 2 = (ax - 1)(x^2 + bx + 2).$$

   What is the value of k?

   (a) 2   (b) 3   (c) -2   (d) -1   (e) 4

19. The integer 29 is the difference of two squares of positive integers. What is the larger of these two integers?

   (a) 14   (b) 11   (c) 13

   (d) 16   (e) 15

20. In how many ways can Yoko, Paul, Ringo, and George be seated in a row of four chairs so that Yoko and Paul don't sit beside each other?

   (a) 12   (b) 4   (c) 16   (d) 8   (e) 20

21. One solution to the equation $x^3 + 2x^2 - 2x - 1 = 0$ is $x = 1$. Which of the following is another solution?

   (a) $\dfrac{1 + \sqrt{5}}{2}$ (b) $\dfrac{-1 + \sqrt{3}}{2}$ (c) $\dfrac{-3 + \sqrt{5}}{2}$

   (d) $\dfrac{-2 + \sqrt{5}}{5}$ (e) $\dfrac{-5 + \sqrt{5}}{2}$

22. Determine which of the following expressions is equivalent to the expression $2x^2+9x-4$.

(a) $2\left(x+\dfrac{9}{4}\right)^2 - \dfrac{145}{16}$

(b) $2\left(x+\dfrac{9}{4}\right)^2 - \dfrac{17}{8}$

(c) $2\left(x+\dfrac{9}{4}\right)^2 + \dfrac{49}{8}$

(d) $2\left(x+\dfrac{9}{4}\right)^2 + \dfrac{17}{16}$

(e) $2\left(x+\dfrac{9}{4}\right)^2 - \dfrac{113}{8}$

23. Given that $i^2 = -1$, simplify the expression $\dfrac{-4+i}{1+4i}$.

(a) i

(b) $-\dfrac{8}{17} + i$

(c) $-i$

(d) $\dfrac{8}{17} - i$

(e) none of these

24. Find a formula for the inverse function of $f(x) = 2 - x$.

(a) $f^{-1}(x) = \dfrac{1}{2-x}$

(b) $f^{-1}(x) = 2 + x$

(c) $f^{-1}(x) = -2 - x$

(d) $f^{-1}(x) = x - 2$

(e) $f^{-1}(x) = 2 - x$

25. Solve the equation $\dfrac{1}{x-2} = \dfrac{3}{x+2} - \dfrac{6x}{x^2-4}$.

(a) -2

(b) 2

(c) 0

(d) 3

(e) no solution

26. Find the area (in square units) of the region bounded by the graphs of $f(x) = |x-3|+1$ and $g(x) = 5$.

(a) $\dfrac{343}{3}$

(b) $\dfrac{11}{4}$

(c) 12

(d) 16

(e) $\dfrac{33}{2}$

27. Jamie bought five copies of her favorite book, one for herself and four as gifts. The books were on sale at 10% off the usual price. After paying tax of 3.8% plus a gift wrapping charge of fifty cents per book, she spent $95.19. What was the original price of the book?

(a) $31.75   (b) $19.95   (c) $13.80   (d) $27.65   (e) $18.00

28. Six machines can manufacture twelve widgets in 6 hours. How many widgets can three of these machines manufacture in nine hours?

(a) 18   (b) 6   (c) 9   (d) 10   (e) 8

29. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers satisfying the relation $a_{n+1} = |a_n| - a_{n-1}$ for every integer $n \geq 2$. If $a_1 = 5$ and $a_2 = -3$ determine a_{2000}.

(a) -3   (b) 7   (c) 2   (d) 5   (e) 3

30. If f is a function with the properties that $f(-x) = -f(x)$ for all x, f is increasing on the interval $3 \leq x \leq 7$, and the minimum of $f(x)$ on the interval $3 \leq x \leq 7$ is 5, then what can be said about f on the interval $-7 \leq x \leq -3$?

(a) f is increasing and $\text{Min}_{-7\leq x\leq -3}\ f(x) = -5$
(b) f is increasing and $\text{Max}_{-7\leq x\leq -3}\ f(x) = -5$
(c) f is decreasing and $\text{Min}_{-7\leq x\leq -3}\ f(x) = -5$
(d) f is decreasing and $\text{Max}_{-7\leq x\leq -3}\ f(x) = -5$
(e) none of these

31. A personnel director selects 3 employees for a certain job from a group of 10 employees. Four of the employees are female and six are male. Determine the probability that exactly two females are selected.

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{2}{10}$ (d) $\frac{5}{7}$ (e) $\frac{3}{10}$

32. If the roots of $x^2 - bx + c = 0$ are $\sin\theta$ and $\cos\theta$ for some real number θ, then what is the value of b^2?

    (a) c (b) $1 + c$ (c) $1 + 2c$ (d) $1 - c$ (e) c^2

33. Let $0 < \theta < \pi/4$. Which one of the following is largest?

    (a) $\sin\theta$ (b) $\dfrac{1}{\sin\theta}$ (c) $\cos\theta$

    (d) $\cot\theta$ (e) $\dfrac{1}{\cos\theta}$

34. The perimeter of a right triangle is $12 + 8\sqrt{3}$ units. The sum of the squares of all three sides is 294 square units. Find the area of this triangle in square units.

    (a) $11 + \sqrt{3}$ (b) 12 (c) $7\sqrt{3}$

    (d) $6\sqrt{3}$ (e) $5 + 4\sqrt{3}$

35. Suppose $p(x)$ is a polynomial with integer coefficients. If 3 and 13 are both roots of p, then which of the following can possibly be the value of $p(10)$?

    (a) 14   (b) 30   (c) 3   (d) 10   (e) 42

36. Which of the following numbers is the largest?

    (a) 6^{1000} (b) 5^{2000} (c) 4^{3000} (d) 2^{5000} (e) 3^{4000}

37. For what values of a will the quadratic equation $(a + 3)x^2 - ax + 1 = 0$ have two distinct real solutions?

    (a) for $a \in (-2, 6)$

    (b) for $a \in (-\infty, 0) \cup (7, \infty)$

    (c) for $a \in (-\infty, -2) \cup (6, \infty)$

    (d) for $a \in (0, 7)$

    (e) none of these

38. Let x be the smallest integer larger than three that has a remainder of 3 when divided by 5 and 6. What is the sum of the digits of x?

    (a) 33   (b) 8   (c) 6

    (d) 14   (e) none of these

39. Find the length of the major axis of the ellipse $x^2 + 5y^2 = 5$.

    (a) 10   (b) $\sqrt{5}$ (c) 5   (d) $2\sqrt{5}$ (e) $\dfrac{2}{5}$

40. In how many of the 120 arrangements of the letters a, b, c, d, and e does b immediately follow a?

    (a) 4!   (b) 3!   (c) 5!   (d) 6!   (e) 2!

41. Determine all values of k for which the graphs of $y = kx - 5$ and $x^2 + y^2 = 9$ have no points in common.

(a) $k = 5$

(b) $k = 0$

(c) all k such that $-\frac{7}{3} < k < \frac{7}{3}$

(d) all k such that $-4 < k < 3$

(e) all k such that $-\frac{4}{3} < k < \frac{4}{3}$

42. Find the range of the function $f(x) = x + \sqrt{1 - x^2}$ defined on $[-1, 1]$.

(a) $[-\sqrt{2}, \sqrt{2}]$ (b) $[-1, \sqrt{2}]$ (c) $[-1, 1]$

(d) $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ (e) $[0, \sqrt{2}]$

43. (*) Early in the morning a light post casts a shadow 13 feet in length. Later in the day, when the inclination of the sun is three times as large, the shadow is 3 feet long. How tall is the post?

(a) $\dfrac{13}{3}$ ft

(b) $\dfrac{11}{3}$ ft

(c) 5 ft

(d) 4.5 ft

(e) not enough information

44. The area of a right triangle is 6864 square inches. The ratio of its legs is $143 : 24$. Find the radius of the circle inscribed in this triangle in inches.
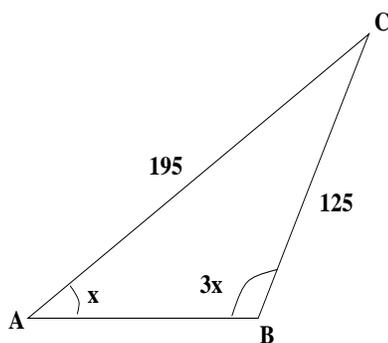
(a) 13   (b) 22   (c) 33   (d) 27   (e) 17

45. (*) For each natural number n, let $S_n = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... + (-1)^{n+1}n^2$. What can be said about the value of $x = S_{1998} + 2S_{1999} + S_{2000}$?

(a) $x = -1$ (b) $x = 1$ (c) $x = 0$

(d) $x = 2000^2$ (e) none of these

46. In the following triangle $\overline{AC} = 195$ inches and $\overline{BC} = 125$ inches. What is $\overline{AB}$?

(a) 111 in   (b) 112 in   (c) 113 in   (d) 100 in   (e) 120 in

47. Given that $\sin \alpha + \sin \beta = \frac{1}{4}$, and $\cos \alpha + \cos \beta = \frac{1}{3}$, determine the value of $\tan(\alpha + \beta)$.

(a) $\frac{10}{27}$ (b) $\frac{5}{27}$ (c) $\frac{8}{27}$

(d) $\frac{24}{7}$ (e) none of these

48. (*) What is the coefficient of $x^5y^2z^2$ in the expansion of $(x + y + 2z)^9$?

(a) 126   (b) 756   (c) 6048

(d) 2016   (e) 3024

49. (*) The following binary code was created by Francis Bacon:

A=aaaa, B=aaaab, C=aaaba, D=aaabb, E=aabaa, F=aabab, ...

What would the code for the letter Z be?

(a) babab   (b) bbbaa   (c) bbbbb   (d) bbaab   (e) ababa

50. (*) The sum of the first three numbers in an arithmetic sequence is 24 and their product is 384. Find the sum of their squares.

(a) 360   (b) 224   (c) 576   (d) 180   (e) 520