ROUND I-SOLUTIONS

1. Solve the equation $\frac{x}{x-3}+\frac{x}{x+3}=\frac{2x+12}{x^2-9}$.

Multiply the equation by x^2-9 to get $x(x+3)+x(x-3)=2x+12$ that is equivalent to $(x+2)(x-3)=0$. From the two solutions $x=-2$ and $x=3$ we observe that $x=-2$ is the only one in the domain of the original equation.

$\boxed{\text{ANSWER. } x=-2}$

2. Which of the numbers 2^{100}, 9^{50}, $\log_4 16^{100}$, and $\left(\sqrt{5}\right)^{100}$ has the largest value?

Notice that $9^{50}=3^{100}$, $\log_4 16^{100}=200$, and that $\sqrt{5}<3$. Therefore the largest value is 9^{50},

$\boxed{\text{ANSWER. } 9^{50}}$

3. What digit is in the 2005th place in the decimal expansion of 2/7?

Observe that $\frac{2}{7}=0.\overline{285714}$ which means that the length of the period is six. On the other hand $2005=334\times 6+1$. The conclusion is that the 2005th place in the decimal expansion of 2/7 is 2.

$\boxed{\text{ANSWER. } 2}$

4. If you select at random a two digit number from 10 to 99, both included, what is the probability that the number is a perfect square or a perfect cube.

The set of perfect squares between 10 to 99, both included, is $S=\{16,25,36,49,64,81\}$ and The set of perfect cubes in the same range is $C=\{27,64\}$. The probability that the number is a perfect square or a perfect cube is given by $P(S\cup C)=P(S)+P(C)-P(S\cap C)$, that is, $P(S\cup C)=\frac{6}{90}+\frac{2}{90}-\frac{1}{90}=\frac{7}{90}$

$\boxed{\text{ANSWER. } \frac{7}{90}}$

5. How many permutations of the letters ABCDEF contain the letters A, B, D adjacent to each other in any order?
   Let us denote by W the group of letters A, B, D. One possible permutation that contains the group W is WCEF. Fixing the group W in the first place gives us 3!=6 permutations of the other 3 letters. Since the three letters in the group W can be rearranged in 3!=6 different possible ways we have a total of 36 permutations of all the letters if we keep that group W in the first place. Since the group W can occupy four different positions, we conclude that the total number of permutations of the letters ABCDEF containing the letters A, B, D adjacent to each other in any order is 144.

$\boxed{\text{ANSWER. } 144}$

6. Let f be a function defined on the set of natural numbers given recursively by $f(1)=1$ and $f(n+1)=f(n)+2^n$ for all $n \geq 1$. Find $f(10)$.

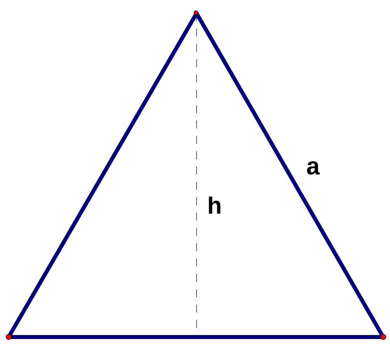
Notice that

$$f(1)=1$$
$$f(2)=2^1+1=3$$
$$f(3)=2^2+3=7$$
$$f(4)=2^3+7=15$$

This suggests that $f(n)=2^n-1$. Thus $f(10)=2^{10}-1=1023$

ANSWER. $2^{10}-1=1023$

7. The area of the equilateral triangle shown in the figure is $\sqrt{3}$ square feet. Find the length of one side.



$h^2=a^2-\frac{a^2}{4}=\frac{3}{4}a^2$, that is $h=\frac{\sqrt{3}}{2}a$; since the area is $\sqrt{3}$ we have that $\sqrt{3}=\frac{1}{2}a\frac{\sqrt{3}}{2}a$. Solving for a we get $a=2$

ANSWER. 2

8. If $\sin\alpha=\frac{1}{2}$ what is the numerical value of $1+\tan^2\alpha$?

Notice that $1+\tan^2\alpha=1+\frac{\sin^2\alpha}{1-\sin^2\alpha}=1+\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{4}{3}$

ANSWER. $\frac{4}{3}$

ROUND II-SOLUTIONS

1. Solve the equation $\sqrt{2x+7}-x=2$.

This equation is equivalent to $\sqrt{2x+7}=x+2$ Now square each side to get $2x+7=(x+2)^2$; this last equation is equivalent to $(x+3)(x-1)=0$ with solutions $x=1$ and $x=-3$. The value $x=-3$ makes the square root negative. The only solution to the original equation is $x=1$.

ANSWER. 1

2. Which of the numbers $3/2$, $\pi/2$, and $\log_4 10$ has the smallest value?

Notice that $\log_4 10 = y$ is equivalent to $10=4^y=2^{2y}$ which implies that $\frac{3}{2}<y$. Therefore the smallest value is $3/2$.

ANSWER. $\frac{3}{2}$

3. If you select a number at random from the first 100 positive integers, what is the probability that the number is divisible by 3 or 4?

The set T of all multiples of 3 that are less than or equal to 100 has 33 elements. The set F of all multiples of 4 that are less than or equal to 100 has 25 elements. The set $T\cap F$ of all multiples of 3 and 4 (multiples of 12) that are less than or equal to 100 has 8 elements. Therefore $P(T\cup F)=\frac{33}{100}+\frac{25}{100}-\frac{8}{100}=\frac{50}{100}=\frac{1}{2}$

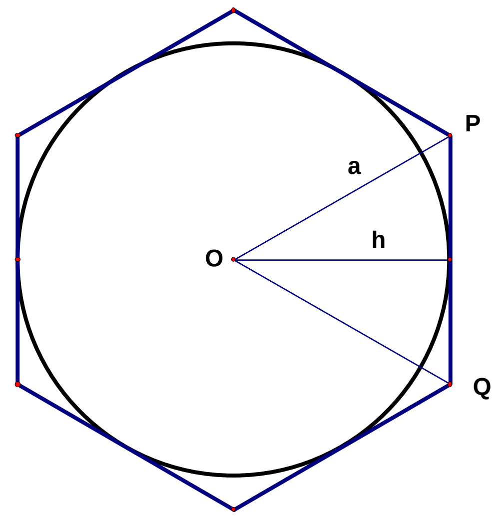
ANSWER. $\frac{1}{2}$

4. When you write the numbers from100 to 1000, how many times do you write the digit 5?

Notice that there are 9 hundreds from 100 to 1000. When writing the 5 in the ones we do it $10\times 9=90$ times. When writing the 5 in tens we do it$10\times 9=90$. When writing the 5 in the hundreds we do it 100 times. All together we write the number five 280 times.

ANSWER. 280

5. The area of the regular hexagon shown in the figure is 12 square feet. Find the area of the circle inscribed in the hexagon.



The equilateral triangle OPQ has area of 2 square feet. That is $2=\frac{1}{2}ah=\frac{1}{2}\frac{2h}{\sqrt{3}}h=\frac{h^2}{\sqrt{3}}$ which means that $h^2=2\sqrt{3}$. The area of the circle is $\pi h^2=2\pi\sqrt{3}$

ANSWER. $2\pi\sqrt{3}$

6. Let f be a function defined for all positive real numbers with the properties that $f(2)=1$ and $f(xy)=f(x)+f(y)$ for all x, y. Find $f(8)$.

Notice that $f(8)=f(2\cdot 4)=f(2)+f(4)=3f(2)=3$

ANSWER. 3

7. A set of fifty numbers has an average of 65. Twenty numbers are discarded from the set so that the average of the remaining numbers is 35. Find the average of the twenty discarded numbers.

Denote by F the sum of the fifty numbers, by W the sum of the twenty discarded numbers and by T the sum of the remaining thirty numbers. Then $F=65\times 50$, $T=35\times 30$ and $\frac{W}{20}=\frac{F-T}{20}=110$

ANSWER. 110

8. Find all values of α such that $0\le\alpha\le\pi$ and $\sin\alpha\cos\alpha=\frac{1}{2}$.

Observing that $\sin\alpha\cos\alpha=\frac{1}{2}\sin 2\alpha=\frac{1}{2}$ we get $\sin 2\alpha=1$ which means $2\alpha=\frac{\pi}{2}$.

ANSWER. $\frac{\pi}{4}$