1) There are 401 gumballs of each color. In the worse possible case the customer could receive (for instance) all 401 red gumballs, then all 401 orange ones, then the 401 yellow ones, then the 401 green ones, before getting 3 blue ones. The conclusion is that the customer needs to buy $4\times 401+3=1607$ gumballs to guarantee he/she gets 3 blue gumballs. ANSWER E) 1607

2) Notice that $(\sqrt[3]{a})^{\log_3 27} = (\sqrt[3]{a})^{\log_3 9^3} = (\sqrt[3]{a})^3 = a$ ANSWER B) a

3) ANSWER C)

4) Notice that $\left(-\frac{1}{27}\right)^{-2/3} = \left(-3^{-3}\right)^{-2/3} = \left(\left(-3^{-3}\right)^{-1/3}\right)^2 = (-3)^2 = 9$. ANSWER D) 9

5) If the reciprocal of $x+a$ is $x-a$ then $x^2-a^2=1$ which means $x=\pm\sqrt{1+a^2}$

ANSWER E) none of these

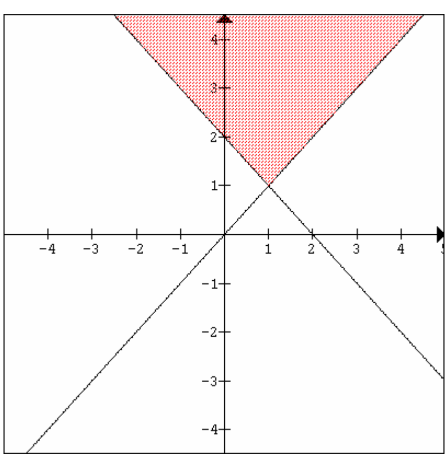
6) Notice that
$(\log_a 3)(\log_3 5)(\log_5 a^4) = (\log_a 3^{(\log_3 5)})(4\log_5 a) = (\log_a 5)(4\log_5 a) = 4(\log_5 a^{(\log_a 5)}) = 4(\log_5 5) = 4$?

ANSWER D) 4

7) $\text{distance} = \text{velocity}\times\text{time} = \frac{a}{6t}\left[\frac{\text{feet}}{\text{secs}}\right]\times 3[\text{minutes}] = \frac{a}{6t}\left[\frac{\text{feet}}{\text{secs}}\right]\times 180[\text{secs}] = \frac{30a}{t}\text{feet} = \frac{10a}{t}\text{yards}$

ANSWER A) $\frac{10a}{t}$ yards

8) The solution set (exclude the boundaries) of $y > x$ and $y > 2-x$ is indicated in the figure.



ANSWER D) I, II

9) Clearly the solution of $\frac{1}{2}\left(\frac{2004}{x-2004}+\frac{2006}{x-2006}\right)=\frac{2005}{x-4010}$ is $x=2005$.

ANSWER E) 2005

10) Notice that by taking log of each side of $x^{\log x}=\frac{x^2}{10}$ we get $(\log x)^2=\log x^2+\log 10$ which is equivalent to $(\log x)^2-2(\log x)+1=(\log x-1)^2=0$. ANSWER B) 10

11) The total area S must be computed by adding the areas of each of the four regions. Notice that the region comprised by the four regions M, N, P, Q is not a right triangle. ANSWER D) $S=32\ in^2$

12) The midpoint of the segment joining the points $P=(3,7)$, $Q=(9,3)$ is given by $M=(6,5)$. The slope of the segment joining the points $P=(3,7)$, $Q=(9,3)$ is $-\frac{2}{3}$. Therefore the equation of the line perpendicular to the segment passing through $M=(6,5)$ is $y-5=\frac{3}{2}(x-6)$

ANSWER A) $y-5=\frac{3}{2}(x-6)$

13) If r is the radius of the first gear and R is the radius of the second one then there is a constant of proportionality k such that $101k=2\pi r$ and $401k=2\pi R$. Therefore one complete turn of the first gear is equivalent to $\frac{101k}{401k}=\frac{2\pi r}{2\pi R}$ turns of the second one, thus if the first one makes 2005 turns per minute, the second one will make $2005\frac{101}{401}=505$ turns per minute. ANSWER B) 505

14) Notice that $\sqrt{3x+7}-x=2$ implies

$$\left(\sqrt{3x+7}\right)^2=(x+2)^2$$

$$0=x^2+x-3$$

which has the two solutions $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{13}$. It is easy to see that the only one satisfying the original equation is $-\frac{1}{2}+\frac{1}{2}\sqrt{13}$. ANSWER C) $-\frac{1}{2}+\frac{1}{2}\sqrt{13}$

15) Notice that $x^4-2x^3-3x^2+2x+2=0$ is equivalent to

$$x^4-2x^3-3x^2+2x+2=0$$

$$x^4-2x^3-2x^2-x^2+2x+2=0$$

$$x^2(x^2-2x-2)-(x^2-2x-2)=0$$

$$(x^2-1)(x^2-2x-2)=0$$

Since the zeros of x^2-2x-2 are irrational numbers, the product of the rational solutions is $(1)(-1)$

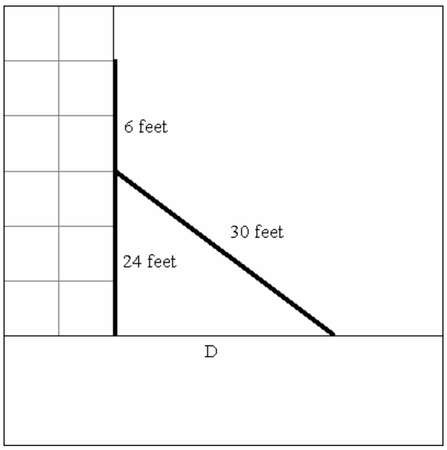
ANSWER D) -1

16) * Notice that $2^{2x} - 3^{2y} = 2005$ is equivalent to $\left(2^x - 3^y\right)\left(2^x + 3^y\right) = 401 \times 5$. Since we are looking for positive integers x, y we conclude that both factors on the left must also be positive integers (because $2^x + 3^y > 0$ for all x, y). This means that we can only have the following two possibilities:

$$\begin{matrix} 2^x - 3^y = 401 \\ 2^x + 3^y = 5 \end{matrix} \quad \text{or} \quad \begin{matrix} 2^x - 3^y = 5 \\ 2^x + 3^y = 401 \end{matrix}$$

Either one of these system implies that $2^x = 203$ which is impossible for integer values of x. (Notice that 2^x is always even for any positive integer x) ANSWER A) 0

17) The figure shows the position of the beam before and after it slipped down 6 feet.



$D^2 = 30^2 - 24^2$

Then $D = 18$

ANSWER A) 18 feet

18) If C denotes the unknown number of pounds of cashews to be used in the mixture, then the revenue from selling $40 + C$ pounds of mixture at \$4 per pound is $4(40 + C)$. This revenue should equal $40 \times 2.5 + C \times 6$. Solving for C in the equation $4(40 + C) = 40 \times 2.5 + C \times 6$ we get $C = 30$

ANSWER A) 30

19) Notice that

$$\cos\theta \cos 2\theta = \frac{1}{4}$$

$$\sin\theta \cos\theta \cos 2\theta = \frac{1}{4}\sin\theta$$

$$\sin 4\theta = \sin\theta$$

Since $0^\circ < \theta < 90^\circ$ we have $90^\circ < 4\theta < 180^\circ$ which means that $\theta = 180 - 4\theta$, that is $\theta = 36^\circ$.

ANSWER D) $30^\circ < \theta < 40^\circ$

20) For any triangle with sides a,b,c its area is given by the formula $A=\sqrt{s(s-a)(s-b)(s-c)}$ where $s=\frac{1}{2}(a+b+c)$. The assumption in this case means that $a=3k$, $b=2k$ and $c=2k$ for some positive real number k. To compute k we observe that $s=\frac{7}{2}k$ and that

$$200^2=\frac{7}{2}k\left(\frac{7}{2}k-3k\right)\left(\frac{7}{2}k-2k\right)\left(\frac{7}{2}k-2k\right)$$
$$200^2=\frac{7}{4}k^4\left(\frac{3}{2}\right)^2$$

Finally $k=10.04$

ANSWER A) 30.12, 20.08, and 20.08

21) The sum of 401 consecutive integers $a_1,a_2,a_3,\ldots,a_{401}$ is equal to 2005. What is a_{200}? If we let $a_1=n+1, a_2=n+2, a_3=n+3,\ldots,a_{401}=n+401$ then $2005=\sum_{i=1}^{401}a_i=\sum_{i=1}^{401}(n+i)=401n+\frac{401\times 402}{2}$. Solving for n we get $n=-196$ which means that $a_{200}=(-196)+200=4$

ANSWER A) 4

22) What is the coefficient of x^0 in the expansion of $\left(x^2+\frac{1}{x^3}\right)^{2005}$ in terms of powers of x? Recall that $(a+b)^n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}a^kb^{n-k}$. If we let $n=2005$, $a=x^2$ and $b=\frac{1}{x^3}$ the generic term looks like $\frac{2005!}{k!(2005-k)!}x^{2k}x^{-3(2005-k)}=\frac{2005!}{k!(2005-k)!}x^{5k-6015}$. Since we are looking for is the coefficient of x^0 we set $5k-6015=0$ to get $k=1203$.

ANSWER A) $\frac{2005!}{802!1203!}$

23) * First notice that $x^{2005}+1=x^{5\times 401}+1=\left(x^5\right)^{401}+1=\left(x^5+1\right)q(x)$ where $q(x)$ is some polynomial in x. If we let $x=2$ then $2^{2005}+1=2^{5\times 401}+1=\left(32\right)^{401}+1=(32+1)q(2)=33q(2)$ which shows that 11 divides $2^{2005}+1$.

ANSWER A) 0

24) The inequality $\left|x^2-3x\right|\le 4x-6$ means that we need to simultaneously consider the following two inequalities:

$$\begin{array}{lcl} x^2-3x\le 4x-6 & & x^2-3x\ge -4x+6 \\ x^2-7x+6\le 0 & \text{and} & x^2+x-6\ge 0 \\ (x-6)(x-1)\le 0 & & (x+3)(x-2)\ge 0 \end{array}$$

The solution set of the first one is $1\le x\le 6$ and the solution set of the second one is $x\le -3$ or $x\ge 2$. The intersection of the two is $2\le x\le 6$.

ANSWER A) $2\le x\le 6$

25) * A zero is produced when a factor 2 and a factor 5 are multiplied. To count the number of times a 5 is repeated we consider the integer part of the quotients $\frac{2005}{5}$, $\frac{2005}{5^2}$, $\frac{2005}{5^3}$, and $\frac{2005}{5^4}$ namely, 401, 80, 16 and 3, respectively. Their sum is 500; this represents the total number of multiples of 5, 25, 125, 625 that are included in the computation of $2005!$. Since there more than 500 factors 2, we conclude that there are 500 zeroes at the end of $2005!$ ANSWER D) 500

26) The amount of salt in the tank at time t is given by $40\frac{\text{grams}}{\text{liter}}\times 20\frac{\text{liters}}{\text{min}}\times t\text{ min} = 800t\text{ grams}$; on the other hand the volume of water in the tank at time t is given by $(4000+20t)$ liters. Then the concentration of salt is $\frac{800t}{4000+20t}$ ANSWER A) $C(t)=\frac{40t}{200+t}$

27) Given a number of the form $a+b\sqrt{2}$, where a,b are rational numbers with $a\neq 0$ or $b\neq 0$, there exists a number $p+q\sqrt{2}$, with p,q rational numbers, such that. What is the value of q?

$\left(a+b\sqrt{2}\right)\left(p+q\sqrt{2}\right)=1$ is equivalent to

$$p+q\sqrt{2}=\frac{1}{a+b\sqrt{2}}=\frac{1}{\left(a+b\sqrt{2}\right)}\frac{a-b\sqrt{2}}{\left(a-b\sqrt{2}\right)}=\frac{a}{a^2-2b^2}-\frac{b}{a^2-2b^2}\sqrt{2}$$

ANSWER C) $\frac{-b}{a^2-2b^2}$

28)

$a_0=3$

$a_1=7$

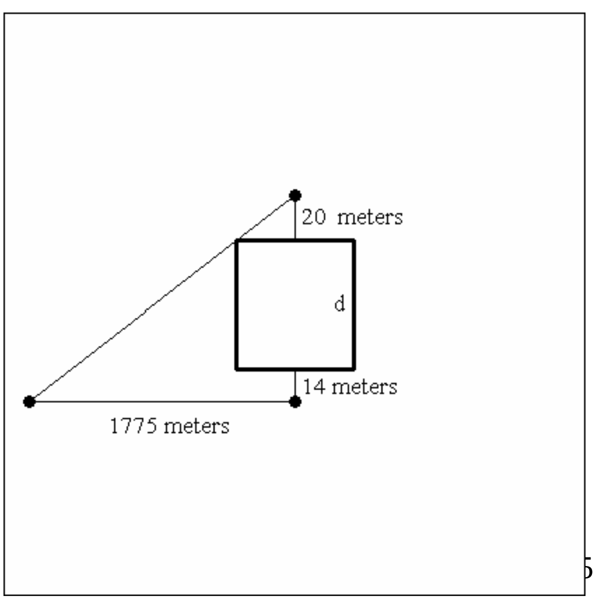
$a_2=3(7)-2(3)=15$

$a_3=3(15)-2(7)=31$

This pattern suggests that $a_n=2^{n+2}-1$. That this is the case can be proved by induction.

ANSWER C) $a_n=2^{n+2}-1$

29)

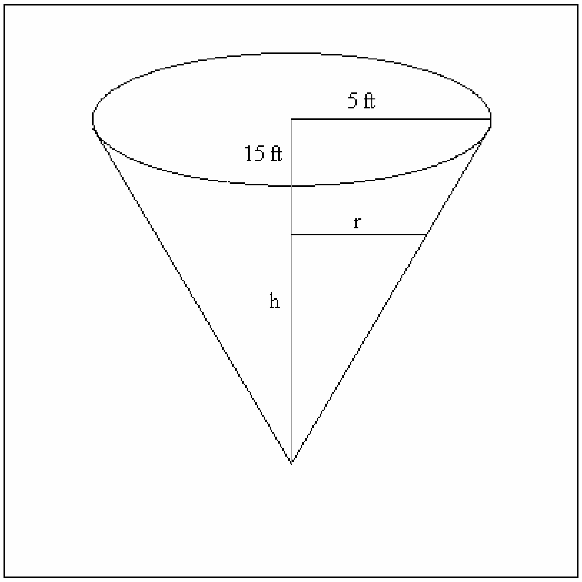


By similarity we have $\frac{20}{d/2}=\frac{20+d+14}{1775}$. This is equivalent to

$$d^2+34d-71000=0$$
$$(d+284)(d-250)=0$$

ANSWER D) 250 meters $\times$ 250 meters

30)



The volume of water at time t is $V(t)=45+3t$. This volume occupies a cone of height $h(t)$ and radius r and therefore $45+3t=\frac{1}{3}\pi r^2 h$; on the other hand, by similarity we have $\frac{r}{h}=\frac{5}{15}$. By substitution we get

$45+3t=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$ which implies that

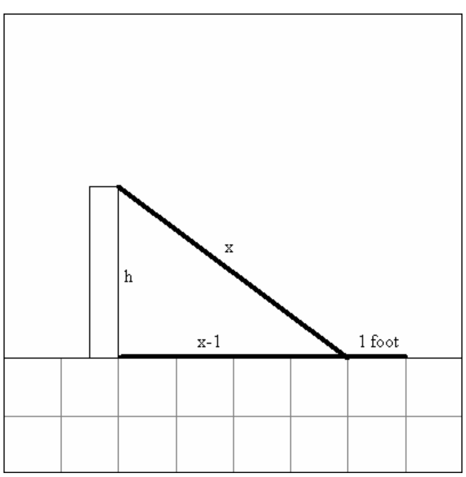
ANSWER A) $h(t)=3\left(\frac{45+3t}{\pi}\right)^{1/3}$

31) Find all values of a for which the equation $x^2-2ax+2a^2-1=0$ has exactly one positive and one negative solution. The product of the two solutions of the equation is $2a^2-1$. Therefore one solution is negative and one is positive if $2a^2-1<0$ which implies $-\frac{\sqrt{2}}{2}<a<\frac{\sqrt{2}}{2}$

ANSWER E) $-\frac{\sqrt{2}}{2}<a<\frac{\sqrt{2}}{2}$

32) * At $t=0$ $A=300$, after 1 period of 3 hours $A=300\frac{2}{3}$ (notice that 2/3 are left), after 2 periods of 3 hours $A=300\left(\frac{2}{3}\right)^2$; in general, after k periods of 3 hours $A=300\left(\frac{2}{3}\right)^k$; Since there are $\frac{t}{3}$ 3-hour periods in t hours, we get that ANSWER B) $A(t)=300\left(\frac{2}{3}\right)^{\frac{t}{3}}$

33) The figure shows that $x^2=h^2+(x-1)^2$. Solving for x we get $x=\frac{h^2+1}{2}$.

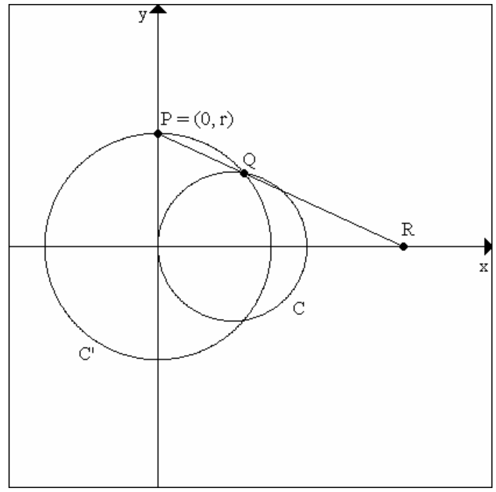


ANSWER A) $\frac{h^2+1}{2}$

34) The intersection of the two circles in the first quadrant is given by the solutions of the system

$$(x-1)^2 + y^2 = 1$$
$$x^2 + y^2 = r^2$$

which can be solved by writing $y^2 = r^2 - x^2$ from the second equation and substituting into the first one to get $Q = \left(\frac{r^2}{2}, \frac{r}{2}\sqrt{4-r^2}\right)$. The coordinates of $P = (0, r)$. If we let $R = (x, 0)$, then using similar triangles we get the relation $\frac{x}{r} = \frac{\frac{r^2}{2}}{r - \frac{r}{2}\sqrt{4-r^2}}$ which simplifies to $x = \frac{r^2}{2-\sqrt{4-r^2}}$. If we rationalize the denominator we get ANSWER B) $2+\sqrt{4-r^2}$



35) Notice that

$$\sqrt{x^2+ax} - \sqrt{x^2+bx} = \frac{\sqrt{x^2+ax} - \sqrt{x^2+bx}}{\sqrt{x^2+ax} + \sqrt{x^2+bx}}\left(\sqrt{x^2+ax} + \sqrt{x^2+bx}\right) = \frac{(a-b)x}{\sqrt{x^2+ax} + \sqrt{x^2+bx}}$$

$$\sqrt{x^2+ax} - \sqrt{x^2+bx} = \frac{(a-b)}{\sqrt{1+a/x} + \sqrt{1+b/x}}$$ (Because $\sqrt{x^2} = x$ for $x > 0$)

Now observe that large values of the quotient are obtained when the denominator is made as small as possible; this requires that we take $x > 0$ to be as large as possible. As x approaches infinity we see that the quotient will never be greater than $\frac{a-b}{2} = 1$ ANSWER C) 1

36) * $g(f(x)) = -2 + \sqrt{f(x)+1} = -2 + \sqrt{(x+2)^2 - 1 + 1} = -2 + \sqrt{(x+2)^2} = -2 + |x+2|$. Since we have $x \le -2$ we conclude that $|x+2| = -(x+2)$ and therefore $g(f(x)) = -2 + |x+2| = -4 - x$

ANSWER A) $-x-4$

$\boxed{37)}$ If we let $\alpha = \sin^{-1}(x)$ then $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and $\sin(\alpha) = x$ (also notice that $\cos(\alpha) > 0$). We this in mind we write $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{\sin(\alpha)}{\sqrt{1-\sin^2(\alpha)}} = \frac{x}{\sqrt{1-x^2}}$ ANSWER E) $\frac{x}{\sqrt{1-x^2}}$

38) * Let us denote by W the word BAD. One possible permutation that contains the word W=BAD is WCEFG. This means that in order to count all the permutations with the word BAD in them we need to count all the permutations of the letters W, C, E, F, and G. This count yields 5! = 120

ANSWER A) 120

39) Two sets U,V are said to be equal if they have the same elements. This says that U,V are equal if

(Every element in U is also in V) *AND* (Every element in V is also in U)

The negation of this statement is

(There is an element in U which is not in V) *OR* (There is an element in V which is not in U)

ANSWER D) IV

40) Notice that $8t+7=8t+4+3=4(2t+1)+3$. This proves that every number in S is also a number of T. Therefore $S \subseteq T$. Now notice that 11 is in T because $11=4(2)+3$ but 11 is not in S because in order to have $11=8t+7$ we must have $t=\dfrac{1}{2}$ which is not an integer.

ANSWER B) $S \subseteq T$

41) We will take the interval $[0,1]$ as the segment of length 1. Let us denote by x the selected point on the segment. The ratio of the shorter to the longer segment is $\dfrac{x}{1-x}$ or $\dfrac{1-x}{x}$ depending on the value of x. These ratios are less than if $\dfrac{x}{1-x}<\dfrac{1}{4}$ or $\dfrac{1-x}{x}<\dfrac{1}{4}$, that is, if $x<\dfrac{1}{5}$ or $\dfrac{4}{5}<x$. Since all points are equally likely to be selected, the probability that the ratio of the shorter to the longer segment is less than $\dfrac{1}{4}$ is $\dfrac{1}{5}+\dfrac{1}{5}$

ANSWER A) $\dfrac{2}{5}$

42) * The total number of different bouquets of half a dozen flowers can be counted explicitly or we can reason in the following way. Consider six slots separated by movable separators S and T with S always to the left of T. One possible arrangement is

_ _ _ S _ _ T _

Let us we agree that to the left of S we put roses (or no roses if no slots are available to the left of S), between S and T we put carnations (or no carnations if S and T are together) and to the right of T we put lilies (or no lilies if no slots are available to the right of T). The problem of counting the total number of different bouquets reduces to counting in how many different ways we can arrange the six _ and the S and T (keeping S to the left). This number is given by ANSWER A) $\dfrac{8!}{2!6!}$

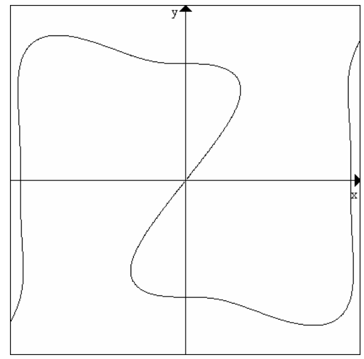
43) The number of committees in which neither one of them serves is $\dfrac{12!}{5!7!}$. If David is in the committee then the other four members must be selected from a group of 12 persons (Juan is out); in this case the number of committees is $\dfrac{12!}{4!8!}$. If we interchange David and Juan we get $\dfrac{12!}{4!8!}$ additional committees.

ANSWER D) $2\cdot\dfrac{12!}{4!8!}+\dfrac{12!}{5!7!}$

44) If we denote by R a motion to the right and by U a motion in the upward direction a generic route from A to B will consists of eight letters chosen from the two letters R or U. For instance, one route could be RURRRURU. Notice that we must always have five R's and three U's. In order to count all different routes from A to B we must count all possible arrangements of five R's and three U's. This number is

ANSWER C) $\dfrac{8!}{3!5!}$

45) The easiest way to solve this problem is by taking advantage of the fact that the graph of the equation looks very linear near the point $(0,0)$. The tangent line must have the form $y = mx$. By plugging in this value of y into the equation we get, after cancellation, $m^3\left(m^2x^2 - 1\right) = \left(x^2 - 2\right)$ (for this we are we assuming $x \neq 0$). As $x \to 0$ the graph of the equation is essentially the graph of the tangent line; in the limit, we get $m^3 = 2$.



ANSWER D) $\sqrt[3]{2}$

46) The set of all pairs is given below

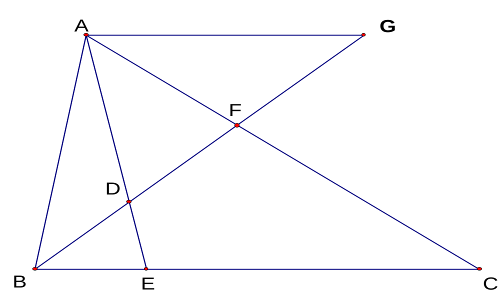
(1,1) (1,2) (1,3) (1,4) (1,5)
(2,1) (2,2) (2,3) (2,4) (2,5)
(3,1) (3,2) (3,3) (3,4) (3,5)
(4,1) (4,2) (4,3) (4,4) (4,5)
(5,1) (5,2) (5,3) (5,4) (5,5)

A symmetric set is symmetric with respect to the main diagonal of the table given above. This means than we can count symmetric sets by counting those sets which contain only pairs of the form (a,b) with $a \leq b$. There are 15 pairs with this property. To build a symmetric set we go over each these pairs and decide whether we include the pair or not. (Two possible values YES or NO for each pair)) This means that there are 2^{15} possible symmetric sets. Among these possibilities with have the case that none of the pairs were selected; this would give us the empty set. By excluding this case we get a total of $2^{15} - 1$ ANSWER B) $2^{15} - 1$

47) First join the midpoints of each side with segments. This way the original triangle is divided into four equal equilateral triangles. In selecting five interior points we always have that at least two of them will belong to the same smaller triangle. So the probability that at least two of the selected points are at a distance less than $1/2$ foot is 1 ANSWER A) 1

48) * Draw a line through A parallel to $\overline{BC}$. Triangle $\triangle BDE$ is similar to triangle $\triangle ADG$ and triangle $\triangle AFG$ is similar to triangle $\triangle BFC$. We have $\dfrac{DE}{DA} = \dfrac{2}{5} = \dfrac{BE}{AG}$ which gives us $AG = \dfrac{5}{2}BE$. Also $\dfrac{FA}{FC} = \dfrac{AG}{BC}$, $BC = BE + EC$, $\dfrac{BE}{EC} = \dfrac{1}{3}$. From this we get that

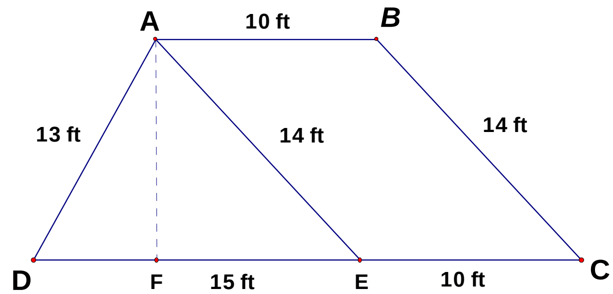
$$BC = BE + 3BE = 4BE \text{ and } \frac{FA}{FC} = \frac{AG}{BC} = \frac{\frac{5}{2}BE}{4BE} = \frac{5}{8}$$



ANSWER A) $\dfrac{5}{8}$

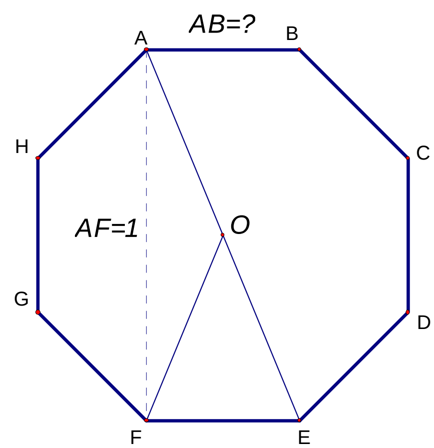
49) Draw a line through A parallel to $\overline{AB}$. The area, 84 ft^2, of the triangle ADE can be computed using Heron's formula. The area of the triangle can be used to get its height AF, namely $AF = \frac{56}{5}$ ft.

The area of the trapezoid is the sum $84 + 10\frac{56}{5} = 196$



ANSWER A) 196 ft^2

50) Notice that $m\angle FOE = \frac{\pi}{4}$. This means that $m\angle FAE = \frac{\pi}{8}$. We then have $\frac{FE}{AF} = \tan\frac{\pi}{8}$, that is $FE = \tan\frac{\pi}{8} = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \sqrt{2}-1$



ANSWER A) $\sqrt{2}-1$